

# Parameterized complexity of fair deletion problems

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# Deletion problems

Given a graph property  $P$  and graph  $G$ , vertex (edge) deletion problem is a task of finding set of vertices (edges) such that  $G$  after removing this set satisfies  $P$ .

## Examples:

- VERTEX COVER –  $W \subseteq V$  such that  $G \setminus W$  has no edge
- FEEDBACK VERTEX SET –  $W \subseteq V$  such that  $G \setminus W$  is a forest
- FEEDBACK ARC SET –  $F \subseteq E$  such that  $G \setminus F$  is a DAG
- ODD CYCLE TRANSVERSAL –  $W \subseteq V$  such that  $G \setminus W$  is a bipartite
- ODD EDGE CYCLE TRANSVERSAL –  $F \subseteq E$  such that  $G \setminus F$  is a bipartite

For monotone properties finding any set is trivial.

We usually aim to find **smallest** such set.

We usually aim to find **smallest** set  $S$  such that  $G \setminus S$  satisfies  $P$ .

In **Fair** deletion problems, we want to find set that is **locally** small.

- For a set  $F$  of edges, we want to minimize  $\deg(F)$
- For a set  $W$  of vertices, we want to minimize  $|N(v) \cap W|$

Arises naturally in various context, e.g. **defective coloring**

# Graph properties

We study properties definable in **Monadic second order logic**

In MSO, we can use

- logical connectives
- quantification over elements
- quantification over **sets**

$(MSO_1)$

Graph is described by set of vertices and adjacency relation  $adj(\cdot, \cdot, )$

$(MSO_2)$

Graph is described by set of vertices and edges and incidence relation  $inc(\cdot, \cdot, )$

# Generalization of deletion problems

Sometimes we want to put additional restriction on the deleted set itself (for example `CONNECTED VERTEX COVER`)

One possibility: MSO formula with one free set variable, goal is to find set  $S$  such that  $G \models \varphi(S)$  (in contrast to original  $G \setminus S \models \psi$ ).

In this way, we can describe

- properties that  $S$  has to satisfy
- properties that  $G \setminus S$  has to satisfy
- and more...

Still called deletion problem by some authors, though **monadic second order evaluation** is more appropriate.

# Parameterized complexity

In parameterized complexity in addition to the input, we have a number called **parameter**.

Examples of parameters:

- size of the solution
- **structural parameters** (treewidth, cliquewidth, size of minimum vertex cover...)

The running time is described as a function of both the size of the input and the parameter.

## Complexity classes

- FPT (**Fixed parameter tractable**) – class of problems solvable in time  $f(k)n^c$
- XP – class of problems solvable in time  $n^{f(k)}$
- W[1]-hard

- XP algorithm for generalized version of FAIR MSO<sub>2</sub> EDGE DELETION PROBLEM by Kolman, Lidický, and Sereni (running time  $f(|\varphi|)n^{O(\text{tw}(G))}$ ).
- Can be easily adapted to vertex version.

MSO FAIR DELETION is  $W[1]$ -hard with respect to treewidth.



# Hardness result for fair deletion

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**FO FAIR DELETION** is  $W[1]$ -hard with respect to **pathwidth + feedback vertex set size**.

It is known that MULTICOLORED CLIQUE cannot be solved in  $f(k)n^{o(k)}$  unless Exponential Time Hypothesis fails (Chen et al. 2006).

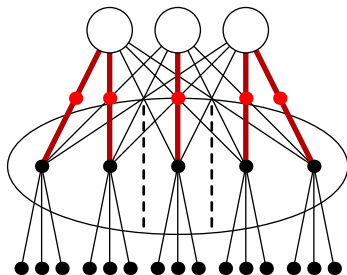
There is no algorithm for FAIR DELETION running in time  $f(|\psi|, k)n^{o(\sqrt{k})}$  where  $k$  is the combined parameter  $(\text{tw}(G) + \text{fvs}(G))$  unless Exponential Time Hypothesis fails

# Proof idea (vertex version)

EQUITABLE PARTITION can be reduced to FAIR DELETION  
(pick favorite problem: EQUITABLE COLORING, EQUITABLE CONNECTED PARTITION. . .)

- We want to delete vertices such that they encode a partition of vertices of  $G$ .
- The fair cost will be minimized for equitable partition.

# Proof idea



We want to partition  $V(G)$  into sets  $T_1, \dots, T_k$  such that

- each  $T_i$  is either a clique or an independent set, and
- between  $T_i, T_j$  is either a complete bipartite graph or no edge at all.

The smallest  $k$  such that the partition exists is the **neighborhood diversity** of  $G$  (denoted by  $\text{nd}(G)$ ).

We allow only  $MSO_1$  formulas ( $MSO_2$  model checking is hard)

We build upon the result of Lampis used in  $MSO_1$  model checking algorithm: Vaguely speaking the formula cannot distinguish between large cardinalities (counting 1, 2, 3, many)



- Can the ETH based lower bound be improved to  $f(|\psi|, k)n^{o(k)}$ ?
- Are there other structural parameters where can we obtain FPT algorithm?

Thank you for attention!