Local Dimension of Posets: Lower&Upper Bounds and Removable Theorems

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Dimension of poset P

A realizer is a non-empty set $\mathcal{L} = \{L_i : i \in [d]\}$ of linear extensions of P such that if x is incomparable to y in P, then $x < y$ in some L_i and $x > y$ in some L_j.

The dimension of P is the size of the smallest realizer of P.

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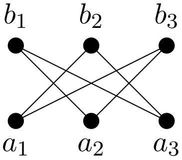

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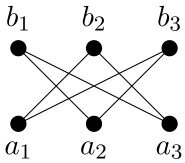

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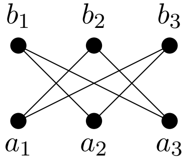

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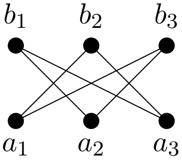

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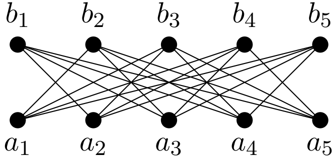
The standard example S_n has $2n$ points and comparabilities $a_i < b_j$ if and only if $i \neq j$.



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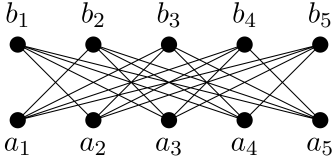

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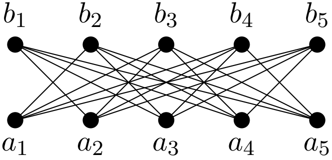

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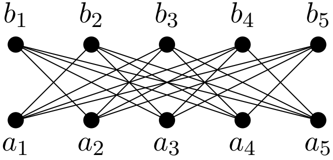

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A local realizer is a non-empty set $\mathcal{L} = \{L_i : i \in [t]\}$ of partial linear extensions (ple) such that

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Observation: Local dimension is at most dimension.

Covering by complete bipartite graphs

Theorem (Erdős and Pyber 1997) Let $G = (V, E)$ be a graph on n vertices. The edge set E can be covered by complete bipartite subgraphs such that each vertex is contained in $O(n/\log n)$ of them.

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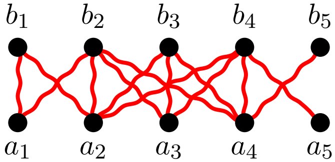
Let P be a height 2 poset with n points. Consider its incomparable pairs.



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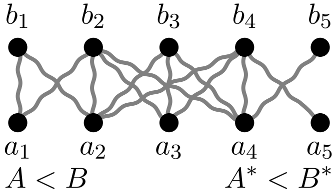

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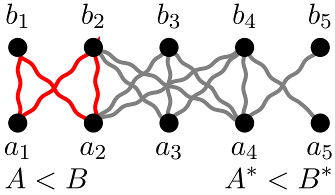

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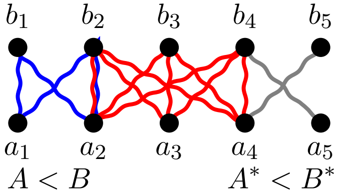

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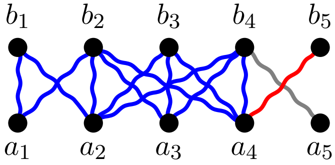

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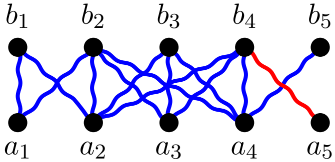

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$b_5 < a_4$ $b_4 < a_5$

Upper bounds

For any poset P, the split Q is poset of height 2 defined as follows:

- Points: $\{x' : x \in P\} \cup \{x'' : x \in P\}$
- Comparabilities: $x' < y''$ in Q if and only if $x \leq y$ in P

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Theorem (Barrera-Cruz, Prag, Smith, Taylor, Trotter)

$$\operatorname{ldim}(Q) - 2 \leq \operatorname{ldim}(P) \leq 2\operatorname{ldim}(Q) - 1.$$

Theorem (KKMSSUW)

For any poset P with n points,

$$\operatorname{ldim}(P) = O(n/\log n).$$

Searching for a lower bound

Theorem (Chung, Erdős, Spencer 1983)

For any $\varepsilon > 0$ and n sufficiently large, there is a graph G such that for any cover of $E(G)$ with complete bipartite graphs $\mathcal{H}$,

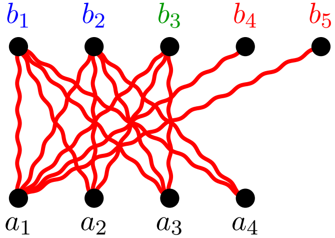
$$\sum_{H \in \mathcal{H}} |V(H)| \geq \frac{(1-\varepsilon)}{2e} \frac{n^2}{\log n}.$$

In particular, there is a vertex that appears in $\Omega\left(\frac{n}{\log n}\right)$ subgraphs.

Difference Graphs

Bipartite graphs of maximum induced matching 1

Equivalently defined as bipartite graphs where vertices of one partite can be ordered such that their neighbours forms an inclusion relation.



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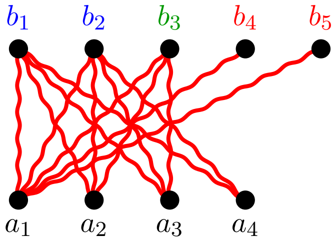

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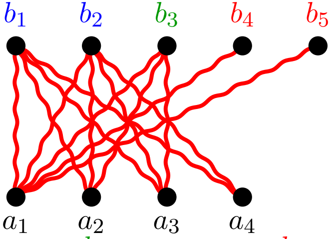

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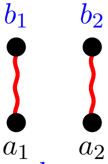

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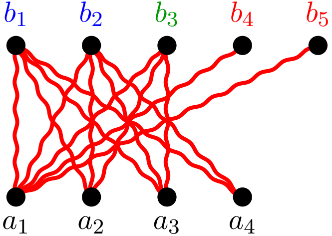

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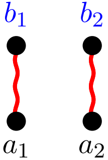

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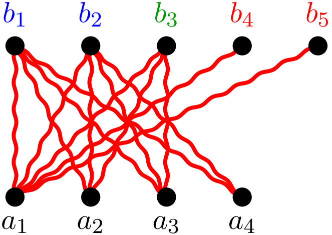

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Theorem (KMMSSUW) For any $\varepsilon > 0$ and n sufficiently large, there is a bipartite graph G such that for any cover of $E(G)$ with difference graphs $\mathcal{H}$,

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Proof idea: Consider a random bipartite graph G with n vertices. Prove that, with high probability, there are no difference graphs H which are subgraphs of G with $|E(H)|/|V(H)|$ "large." If G has enough edges, then the property in the theorem holds.

Difference graphs

$$\text{Type I: } \frac{|E(H)|}{|V(H)|} > \frac{\ln n}{1-\varepsilon} \qquad \text{Type II: } \frac{|E(H)|}{|V(H)|} \leq \frac{\ln n}{1-\varepsilon}$$

Let G be a random bipartite graph with $n/2$ labeled vertices in each partite set and edges included with probability $1/e$.

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Probability that G contains a particular Type I difference graph is at most $2^{-\varepsilon|E(H)|}$.

Probability that G contains any Type I difference graph goes to zero with n going to infinity.

With high probability, a random bipartite graph has no Type I difference graph.

With high probability, a random bipartite graph G has no Type I difference graphs and at least $\frac{n^2}{8}$ edges. Let $\mathcal{H}$ be an arbitrary difference graph cover of G.

$$\begin{aligned}\sum_{H\in\mathcal{H}} |V(H)| &\geq \sum_{H\in\mathcal{H}} \frac{|E(H)|(1-\varepsilon)}{\ln n} \\ &\geq |E(G)|\frac{1-\varepsilon}{\ln n} \\ &= \Omega\left(\frac{n^2}{\ln n}\right).\end{aligned}$$

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So some vertex of G is in $\Omega\left(\frac{n}{\ln n}\right)$ of the difference graphs in $\mathcal{H}$.

Removable theorems

The removable pair conjecture for dimension by Trotter 1971

For any poset P with at least 3 points, there is a pair of points $\{x, y\}$ such that $\dim(P) \leq \dim(P - \{x, y\}) + 1$.

The analogous conjecture can be made for local dimension.

Removable theorems

Theorem (Removable pair for posets of height 2 KMMSSUW)

For a poset $P = (P, \leq)$ with $|P| \geq 3$ and height at most 2, there are two elements x, y in P such that

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An analogous result to a theorem by Tator 94, showing that one can find four elements whose removal decreases the local dimension by at most two.

Theorem (Removable quadruple KMMSSUW)

For any partially ordered set $P = (P, \leq)$, there are $x, y, z, w \in P$ such that

$$\text{ldim}(P) \leq \text{ldim}(P - \{x, y, z, w\}) + 2.$$

Removable quadruple — useful lemma

Theorem (Bogart 72)

Let $P = (X, \leq)$ be a poset, C_a and C_b are chains of P such that each element of C_a is incomparable with each element of C_b, then there is a linear extension L of P

- in which each element of X that is incomparable with an element of C_a is above each element in C_a with which it is incomparable,
- and each element of X that is incomparable with an element of C_b is below each element of C_b with which it is incomparable.

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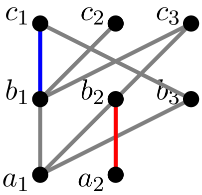

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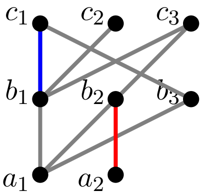

$L: \quad a_1 < b_1 < b_3 < c_1 < c_2 < a_2 < b_2 < c_3$

Removable quadruple — main ingredient

Lemma (KMMSSUW)

If x is a minimal element, y is a maximal element and x, y are incomparable in the partially ordered set $P = (X, \leq)$ for $|X| \geq 3$, then

$$\text{ldim}(P) \leq \text{ldim}(P - \{x, y\}) + 1.$$

- Local realizer $\mathcal{L} = \{L_1, \ldots, L_t\}$ of $P - \{x, y\}$

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- z in $X - \{x, y\}$, then we can assume that $z \in L_i$ for $i \in \{1, \ldots, d\}$.

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Boolean Lattice

Theorem (KMMSSUW)

Let $2^{[n]}$ denote the Boolean lattice of dimension n. There exists a constant c such that

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For any partially ordered set $P = (X, \leq)$ for $|X| \geq 3$, there are two elements x, y in X such that $\mathrm{ldim}(P) \leq \mathrm{ldim}(P - \{x, y\}) + 1$.

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Question

Does there exist a poset P such that no local realizer $\mathcal{L}$ of P of optimal size also contains a linear extension of P?

What about parameters?

Find a good problem such that there is an FPT algorithm parameterized by local dimension.

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Thank you for your attention!