Triangle-free planar graphs with the smallest independence number

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Motivation

(a) 4-color theorem $\Rightarrow$ each planar graph G has $\alpha(G) \geq \frac{n}{4}$

However, characterization of planar graphs such that $\alpha(G) = \frac{n}{4}$ is not known.

(b) It's a Ramsey problem: We are asking (for planar graphs) how large must be $\alpha(G)$ when $\omega(G) < 3$.

(c) [Dvořák and Mnich 2014] found an algorithm that decides whether triangle-free planar graph G has $\alpha(G) \geq \frac{n+k}{3}$ in time $2^{O(\sqrt{k})}n$.

History

Theorem [Grötzsch 1959]:
Every planar triangle-free graph is 3-colorable.

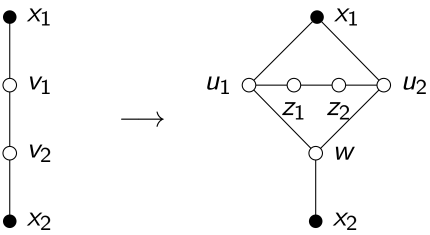
$\Rightarrow$ If G is an n-vertex planar triangle-free graph, then $\alpha(G) \geq n/3$

Theorem [Steinberg and Tovey 1993]:
If G is an n-vertex planar triangle-free graph, then $\alpha(G) \geq (n+1)/3$.

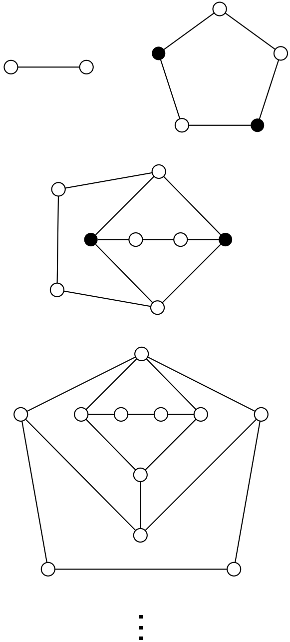
- They also described an infinite class $\mathcal{G}$ of planar triangle-free graphs for which their result is tight.

The structure of the class $\mathcal{G}$

Definition: The class $\mathcal{G}$ consists of the path P_2 on two vertices, the 5-cycle, and all graphs obtained from the 5-cycle by a repeated application of the path–diamond replacement.



Path–diamond replacement.



Our results

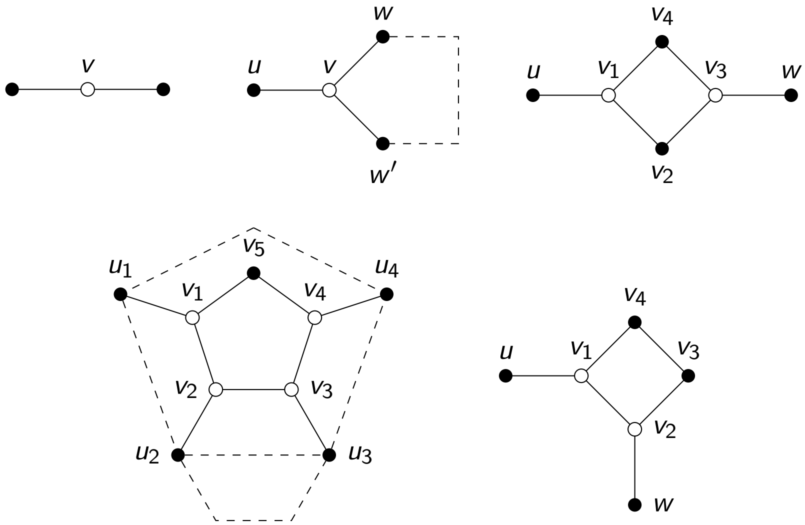
Theorem:
If G is a planar triangle-free graph with n vertices and $G \notin \mathcal{G}$, then $\alpha(G) \geq (n+2)/3$.

- We give a new proof of theorem of Steinberg and Tovey.
- Our proof also implies that $\mathcal{G}$ contains all the graphs for that their bound is tight.

Outline of the proof

1. We describe certain subgraphs as reducible configurations.
2. We prove that a minimal counterexample to our claim can not contain any reducible configuration.
3. By discharging method we prove that every planar triangle-free graph contains some reducible configuration.
4. Steps 2. and 3. yield a contradiction.

Reducible configurations



Excluding the configurations

Definition: An n-vertex graph G is *tight* if G is planar, triangle-free, and $\alpha(G) = (n+1)/3$.

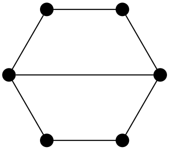
A *minimal counterexample* is a tight graph $G \notin \mathcal{G}$ with the smallest number of vertices.

- Minimal counterexamples do not contain diamonds.
- Minimal counterexamples have minimum degree at least three.
- All reductions preserve tightness, planarity and triangle-free property, hence reduction of minimal counterexample is in $\mathcal{G}$.
- No reduction of such graph could produce graph of $\mathcal{G}$ (key is the placement of degree 2 vertices).

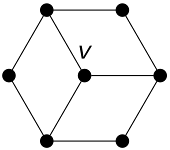
$\Rightarrow$ Therefore minimal counterexample does not contain any reducible configuration.

Unavoidability (1)

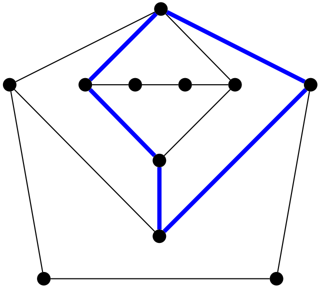
Definition: A cycle C in a plane graph G is *dangerous* if its length is at most 6, C does not bound the outer face of G, and G_C is distinct from C itself, $C_{6,c}$ and $C_{6,v}$.



$C_{6,c}$



$C_{6,v}$



Example of a dangerous cycle

Unavoidability (2)

We will consider a planar triangle-free graph G with the outer face bounded by a (≤ 6)-cycle K, such that G is distinct from K itself, $C_{6,c}$ and $C_{6,v}$.

Theorem: If G does not contain any dangerous cycle, then it contains one of the configurations Conf(1), ..., Conf(5) that does not interfere with the outer face.

If G contains a dangerous cycle, then we use the theorem on $H \subset G$ consisting of that cycle and everything inside. By induction, there is some reducible configuration in H and therefore there is a reducible configuration in G either.

Unavoidability (3) – initial charge

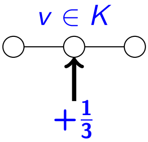
Proof by discharging

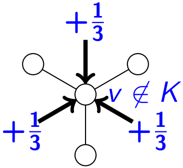
Initial charge:

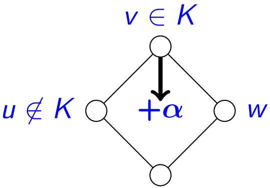
- vertex v gets initial charge $c_0(v) = \deg(v) - 4$
- face f gets initial charge $c_0(f) = |f| - 4$
- by Euler's formula, the sum of initial charges is

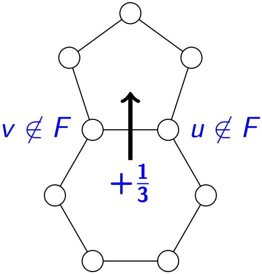
$$\sum_{v \in V} (\deg(v) - 4) + \sum_{f \in F} (|f| - 4) = -8$$

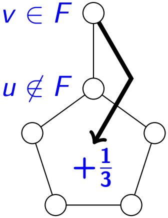
Unavoidability (4) – discharging rules





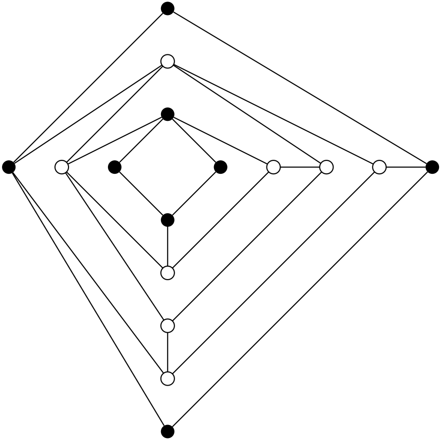






Open problems

Is it true that $\forall k\ \exists n_0$ such that each planar, triangle-free graph G where $n \geq n_0$ and $\alpha(G) \leq (n+k)/3$ contains a following subgraph?



white vertices have no other neighbours