Computational complexity of distance edge labeling

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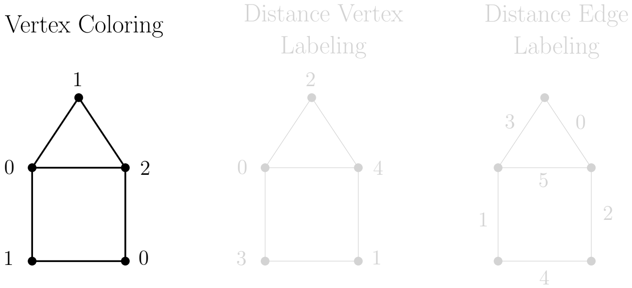
Section 1

Introduction

Graph coloring

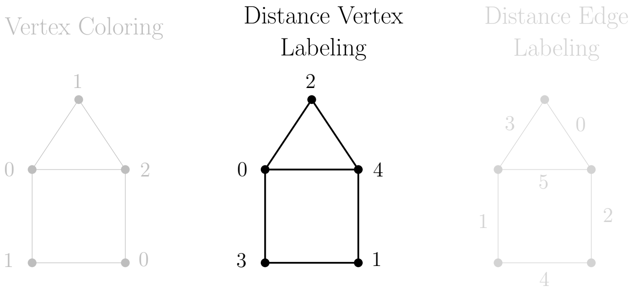


Definition (Vertex coloring mapping)

Vertex-coloring mapping $f\colon V \to \mathbb{N}_0$ satisfies:

- $|f(v) - f(\tilde{v})| \geq 1$ for neighboring vertices $v, \tilde{v}$.

Graph coloring

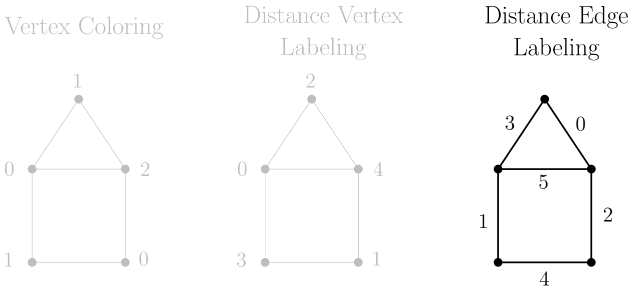


Definition (Vertex labeling mapping)

Vertex labeling mapping $f_{(2,1)} \colon V \to \mathbb{N}_0$ satisfies:

- $|f_{(2,1)}(v) - f_{(2,1)}(\widetilde{v})| \geq 2$ for neighboring vertices $v, \widetilde{v}$.
- $|f_{(2,1)}(v) - f_{(2,1)}(\widetilde{v})| \geq 1$ for vertices $v, \widetilde{v}$ at distance two.

Graph coloring



Definition (Edge labeling mapping)

Edge labeling mapping $f'_{(2,1)} : E \to \mathbb{N}_0$ satisfies:

- $|f'_{(2,1)}(e) - f'_{(2,1)}(\widetilde{e})| \geq 2$ for neighboring edges $e, \widetilde{e}$.
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Vertex problem definition

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Problem (The Distance vertex labeling problem.)

Input:	A graph G and a parameter λ.
Question:	Is $\lambda_{(2,1)}(G) \leq \lambda$?

Previous results

Theorem (The dichotomy of the Distance vertex labeling problem)
(Fiala, Kloks, Kratochvíl 2001)

The Distance vertex labeling problem is:

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Theorem (The NP-completeness of chordal graphs.) (Bodlaender, Kloks, Tan, van Leeuwen 2004)

It is NP-complete to decide whether $\lambda_{(2,1)}(G) \leq n$ for the chordal graph G on n vertices.

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- This version is equivalent to the vertex version on the class of line-graphs.

Our results

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Corollary (An exponential lower bound for the Distance edge labeling problem) (Knop, TM 2016)

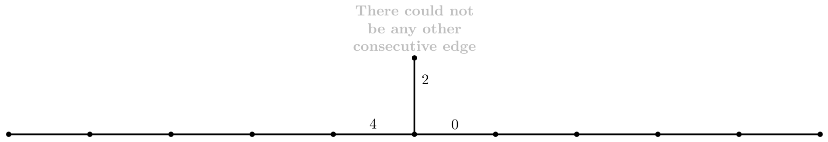
For every fixed span λ fulfilling $\lambda \geq 5$ there is a positive real s such that the Distance edge labeling problem parametrized by its size n cannot be solved in time $2^{sn}n^{\mathrm{O}(1)}$, unless the Exponential Time Hypothesis fails.

Section 2

Polynomial cases

Polynomial cases ($\lambda \leq 4$)

An Example of the case analysis



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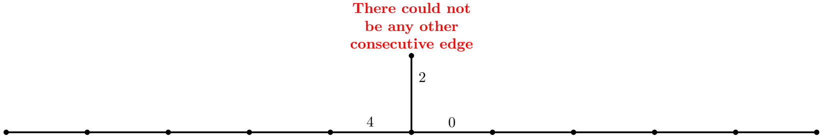
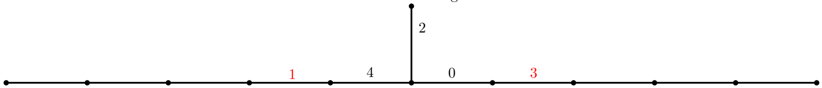


Polynomial cases ($\lambda \leq 4$)

An Example of the case analysis

There could not be any other consecutive edge



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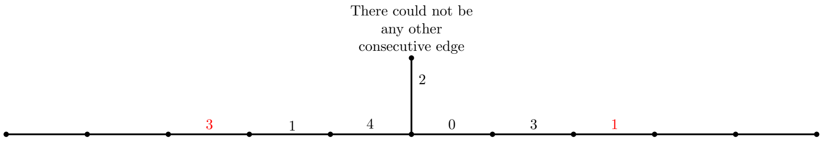

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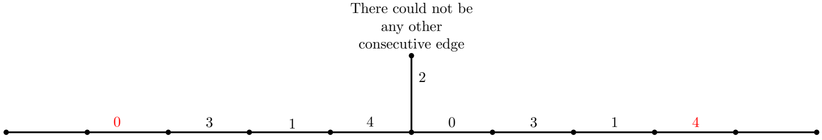

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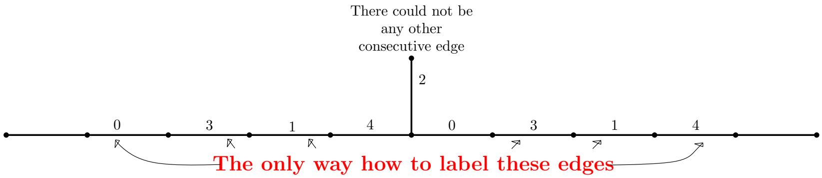

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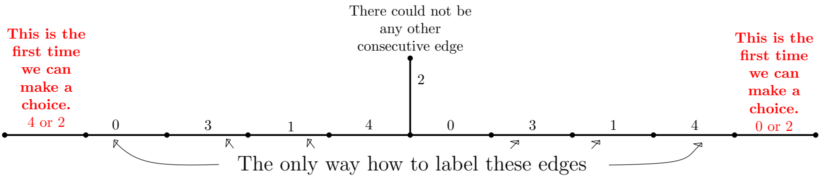

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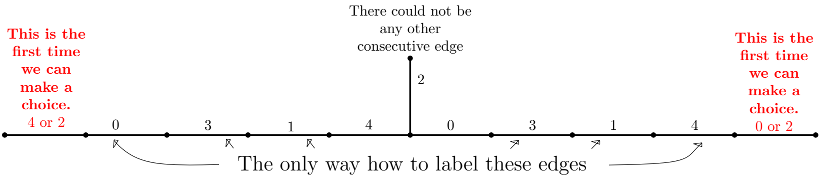

- By a similar analysis the exact characterisation of graphs admitting polynomial-time algorithm is given in the paper.

Polynomial cases ($\lambda \leq 4$)

Characterisation

- All paths and cycles.

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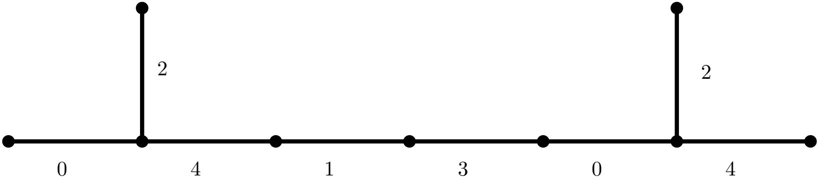
- Some generalised paths according to a parity between vertices of degree three.



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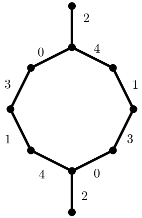

Section 3

NP-completeness

Monotone not all equal 3-SAT

Problem (Monotone not all equal 3-SAT)

Input:	A 3-MCNF formula φ.
Question:	Is it possible to find an assignment such that each clause has at least one literal set to true and at least one literal set to false?

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An Example clause

A clause in 3-MCNF (i.e. 3-CNF without negations):

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee x_4 \vee x_5).$$

And an assignment: $x_1 = T$; $x_2 = F$; $x_3 = T$; $x_4 = F$; $x_5 = T$.

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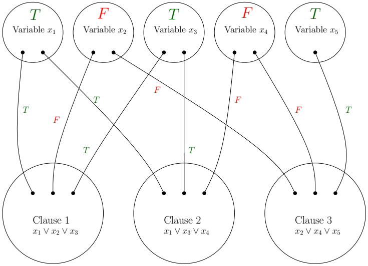
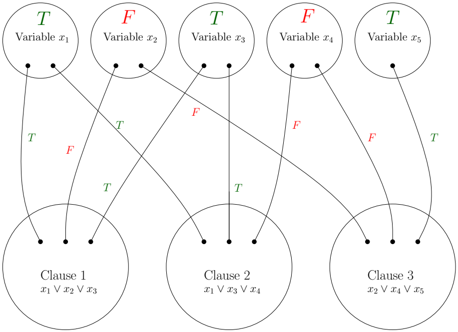


Our reduction

Interesting information about reductions

- Our reduction has five independent parts.



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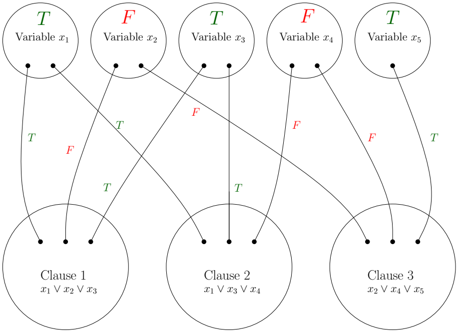

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Interesting information about reductions

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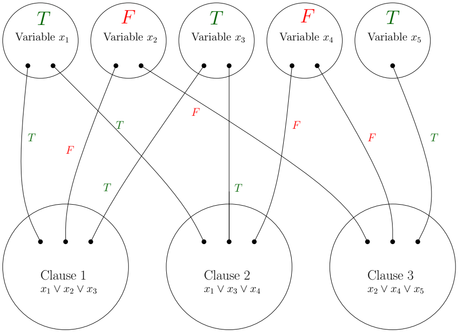
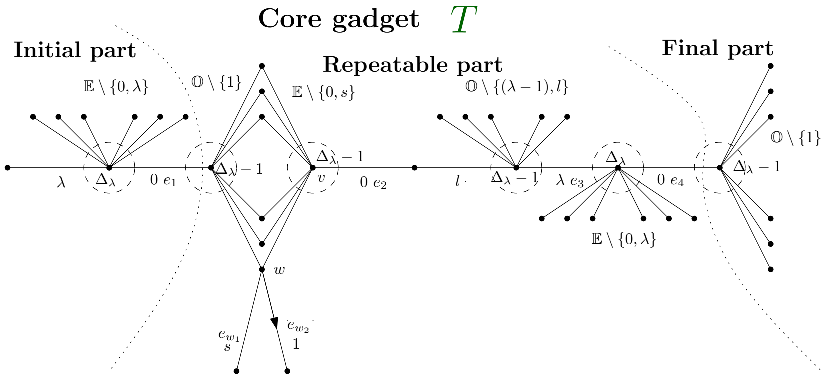
    - It was necessary for us to distinguish cases for even and odd λ.
    - And solve cases for small $\lambda = 5, 6, 7$ by different reductions.



Our reduction for even $\lambda \geq 8$



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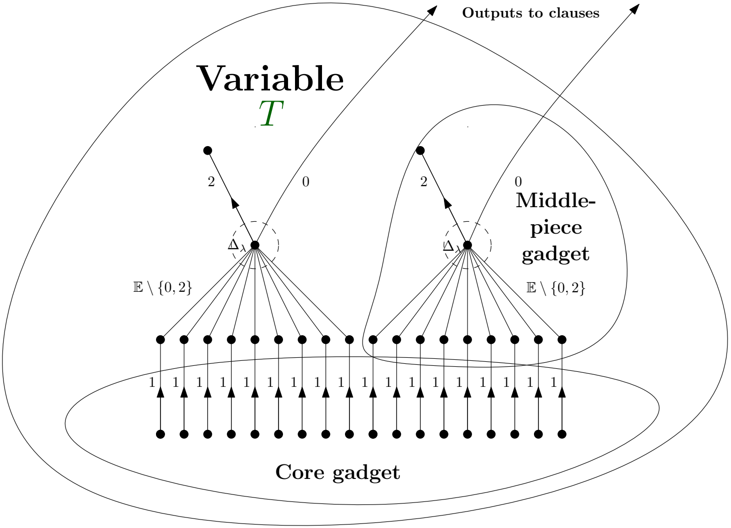

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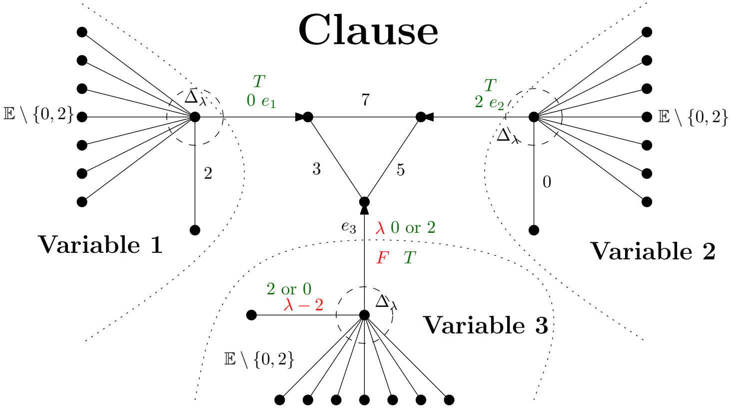

Section 4

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Open problems

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Thank you for your attention!