Complexity of Packing Coloring

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Vysoké Tatry, Slovakia

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Problem definition

Definition (Packing k-coloring (Goddard, Hedetniemi², Harris, Rall 08'))

Given a graph $G = (V, E)$ and integer k, a packing k-coloring is a mapping $c : V \to \{1, \ldots, k\}$ such that any two vertices u, v of the same color $(c(u) = c(v))$ are in distance greater than $c(u)$ $(d(u, v) > c(u))$.

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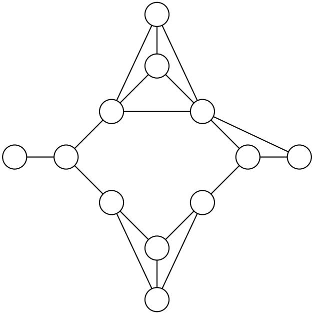

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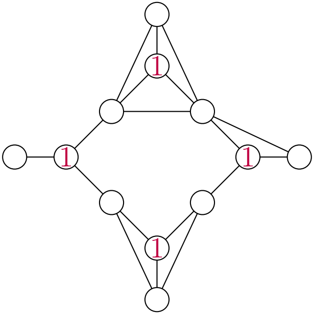

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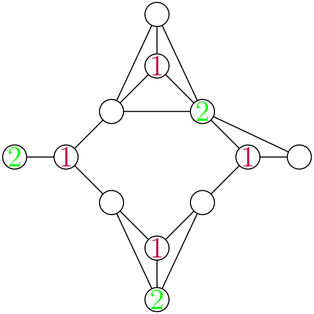

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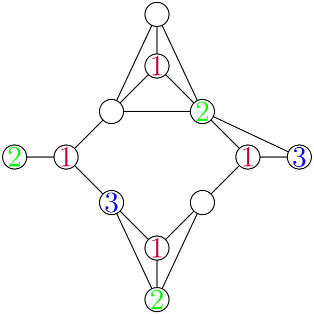

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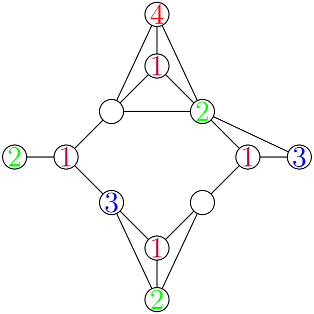

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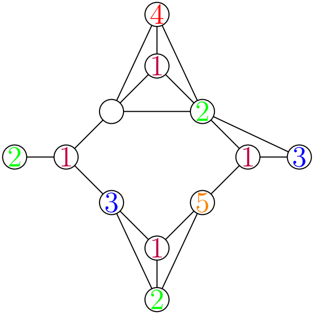

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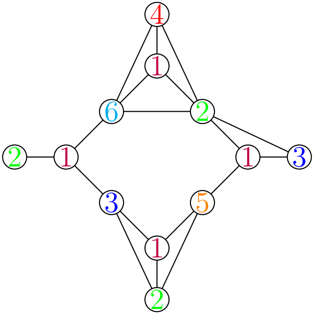

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Observation

In the graph of diamater d only the colors $1, \ldots, d - 1$ can be used more than once.

Known results

[Goddard, Hedetniemi^2, Harris, Rall 08']

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- NP-complete for chordal graphs of diameter at most 5.

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- NP-complete for chordal graphs of diameter at most 5.
- FPT algorithm on chordal graphs parameterized by k.

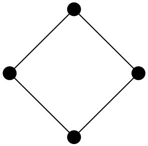
Our result — hardness

Theorem (Hardness on chordal graphs)

Packing chromatic number is NP-complete on chordal graphs of any fixed diameter ≥ 3 on n vertices. Moreover, it is hard to approximate within $n^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$ and any fixed diameter ≥ 3, unless $\mathsf{NP} = \mathsf{ZPP}$.

Hardness construction

(a) Start with graph G, denote the set of its vertices by V,



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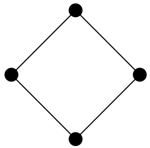
Hardness construction

- (a) Start with graph G, denote the set of its vertices by V,
- (b) subdivide every edge once, denote the set of new vertices by S,



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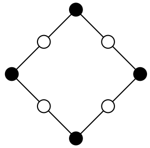

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Hardness construction

- (a) Start with graph G, denote the set of its vertices by V,
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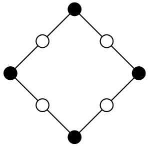
- (c) create a clique from vertices in S,



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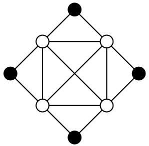

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- (a) Start with graph G, denote the set of its vertices by V,
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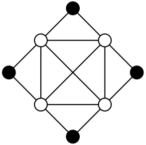
- (d) for every $v \in V$ add a duplicate vertex v' and the edge vv'; denote the set of new duplicate vertices by D,



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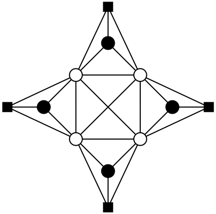

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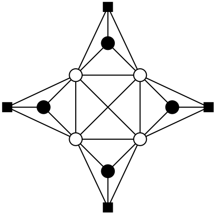
Hardness construction

- (a) Start with graph G, denote the set of its vertices by V,
- (b) subdivide every edge once, denote the set of new vertices by S,
- (c) create a clique from vertices in S,
- (d) for every $v \in V$ add a duplicate vertex v' and the edge vv'; denote the set of new duplicate vertices by D,
- (e) to increase the diameter to $d > 3$, add a path P of length $d - 2$ starting in one vertex in S.



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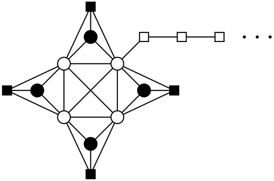

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Let k be fixed and G be an interval graph. Finding a partial coloring by colors $1, \ldots, k$ that is maximizing the number of colored vertices can be solved in XP time ($O(n^{k+1})$).

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Let k be fixed and G be a graph of clique-width q. Finding a partial packing coloring by colors $1, \ldots, k$ that is maximizing the number of colored vertices can be solved in FPT time parameterized by q.

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Packing chromatic number for interval graphs of diameter d can be solved in XP time ($O(n^{d \ln(5d)})$).

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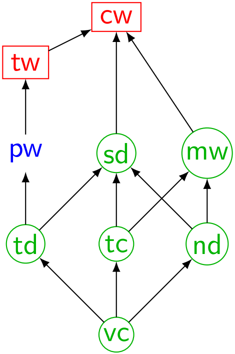

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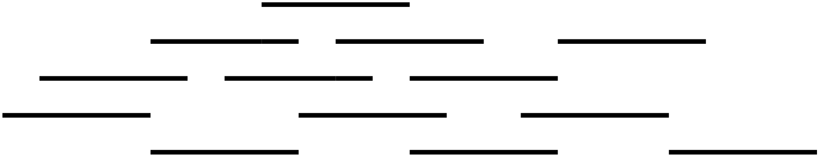
- Since G is perfect we can find a partition into k independent sets $X_1, \ldots X_k$ in polynomial time.



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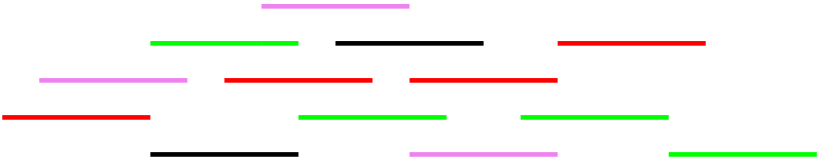

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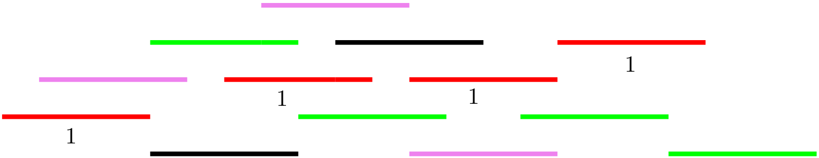

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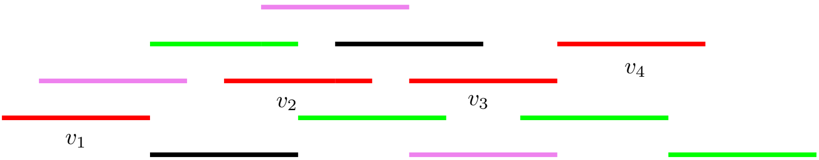

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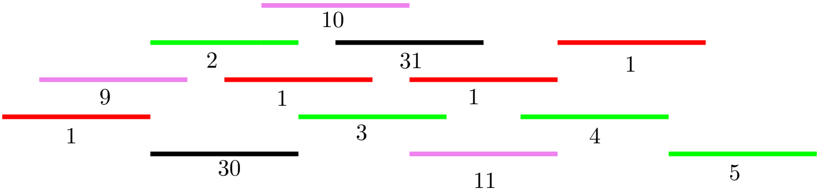

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- We have just obtained a bound on a number of colors in terms of k (the size of the largest clique) so we can use an FPT algorithm by [Fiala and Golovach 10'].

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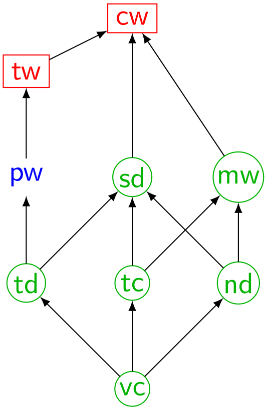

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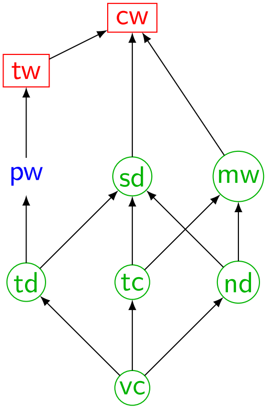

Thank you for your attention!