Robust Connectivity of Graphs on Surfaces

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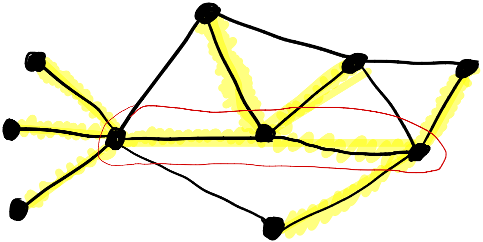
Maximum Leaf Number $\equiv$ Minimum Connected Dominating Set

Definition (Maxleaf Number)

$\Lambda(T)$:= the **set of leaves in a tree T**.
τ_G := the **set of all spanning trees in G**.
The ***maxleaf number*** of a graph G is:

$$\ell(G) := \max_{T \in \tau_G} |\Lambda(T)|.$$



$\ell(G) = 7$

$|CDS(G)| = 3$

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- Garey & Johnson '79: **NP-complete** on planar graphs of max deg 4.
- Storer '81: G **cubic** has $\ell(G) \geq \lceil \frac{n}{4} + 2 \rceil$.
- Kleitman & West '91: Conn. G of **min deg k**: $\ell(G) \geq (1 - \frac{2.51 \ln k}{k})n$.

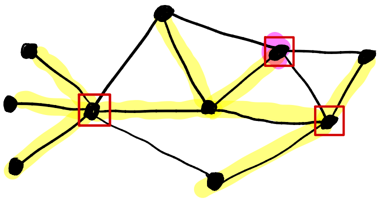
Robust Connectivity—Definition

Definition (Robust connectivity BMS '20)

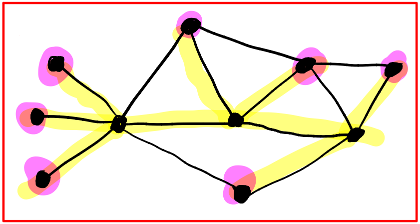
$$\kappa_\rho(G) := \min_{\substack{R \subseteq V(G) \\ R \neq \emptyset}} \max_{T \in \tau_G} \frac{|R \cap \Lambda(T)|}{|R|}. \leq \frac{\ell(G)}{V(G)}$$

$\boxed{R}$

$K_\rho(G) \leq \frac{1}{3}$



$K_\rho(G) \leq \frac{7}{10}$ $R = V(G)$



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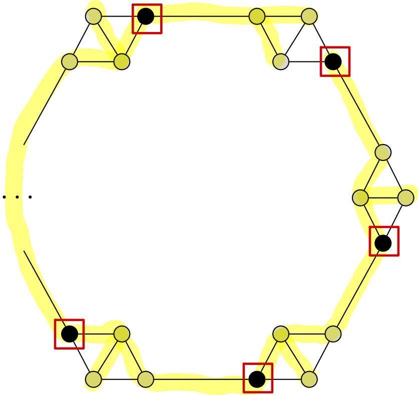

Robust Connectivity—Previous Results

Theorem (BMS '20)

*If $\Delta \geq 3$, then there exists $\varepsilon = \varepsilon(\Delta) > 0$ such that if G is a **3-connected** graph of **maximum degree Δ**, then $\kappa_\rho(G) \geq \varepsilon$.*

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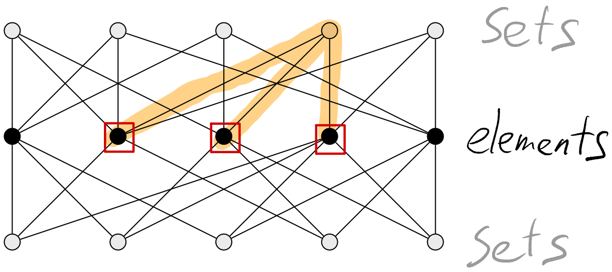
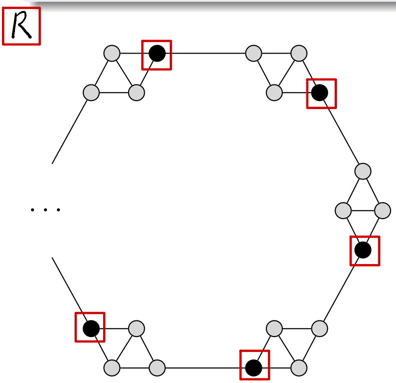


Robust Connectivity—Genus Bound

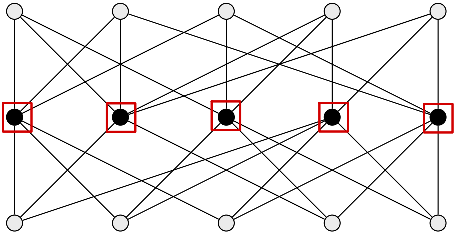
Theorem (BMNS '21)

*If $r \geq 3$ and G is an **r-connected graph of Euler genus γ**, then*

$$\kappa_\rho(G) \geq \frac{1}{27}\gamma^{-1/r}.$$



Tight examples



Albertson & Berman Conjecture '79

Conjecture (Albertson & Berman Conjecture '79)

*If G is a **planar graph** on n vertices, then G contains an **induced forest of size at least** $n/2$.*

Theorem (Borodin '76)

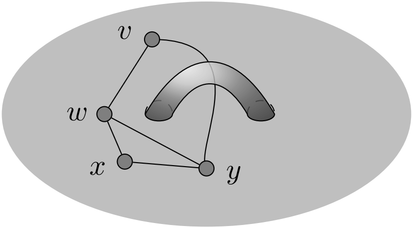
If G is a planar graph on n vertices, then G contains an induced forest of size at least $2n/5$.

Known on Some Subclasses of Planar Graphs

- Hosono '90: Outerplanar.
- Salavatipour '06: Triangle-free planar.
- Kelly & Liu '17 / Shi & Xu '16: Planar of girth 5.

Surface Connectivity

$\tilde{G}$:=Some specific **embedding of G**.
$m(\tilde{G})$:=**# of vertices** in a **largest** induced embedded subgraph $\tilde{G}' \subseteq \tilde{G}$ for which S ✂ $\tilde{G}'$ is a **connected surface**.



Definition (Surface Connectivity)

The ***surface connectivity*** $\kappa_s(S)$ of a surface S is defined as follows:

$$\kappa_s(S) = \inf \left\{ \frac{m(\tilde{G})}{|\tilde{G}|} : \tilde{G} \in \mathcal{G}_S \right\}.$$

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*Let S be a surface. Every graph G with an **edge-maximal** embedding on S satisfies $\kappa_\rho(G) \geq k$ **if and only if** $\kappa_s(S) \geq k$.*

Surface Connectivity—Albertson & Berman Relation

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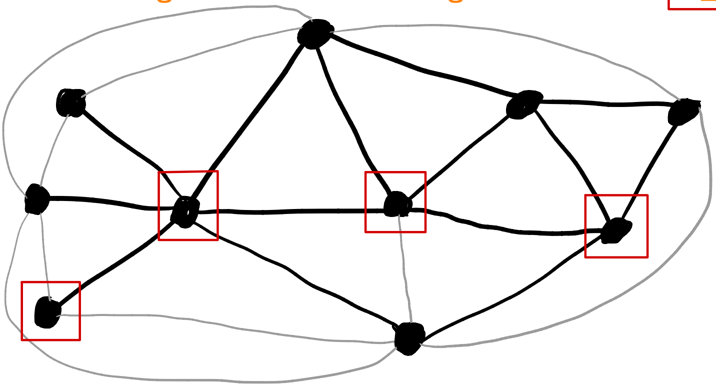

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Lemma

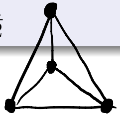
*Let $\tilde{G}$ be an edge-maximal emedding of G in a surface S and let $\tilde{G}' \subseteq \tilde{G}$. If S ✂ $\tilde{G}'$ **is a connected surface**, then $G \setminus G'$ **is a connected**.*

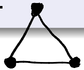
$\tilde{G}' := \{v_i, v_j\}$

S ✂ $\tilde{G}'$ Connected

Easy Cases Implies

$k \leq \frac{1}{2}$ $\quad |V(G)| \geq 3. \quad$ G is 3-connected. $\Longrightarrow |R| \geq 4.$

1 

2 

3

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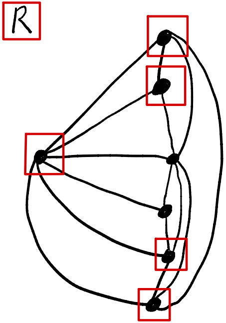
G **has a dominating vertex**



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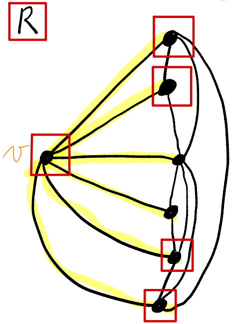

All but v are leaves!

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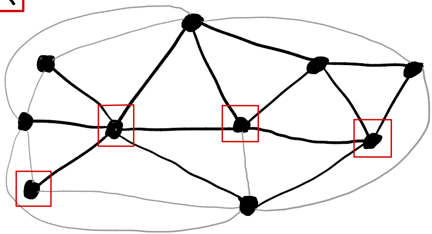
$\boxed{R}$

G **does not have any dominating vertex**



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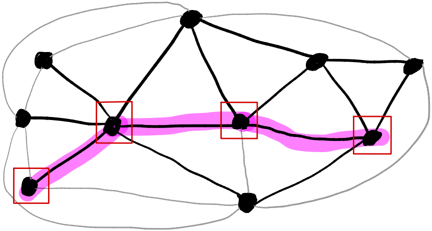

$G[R]$ $\kappa_s(S) \geq k$

$\Downarrow$ 1 $k \leq \frac{1}{2}$ $|R| \geq 3$

$\exists R' \subseteq R$ $|R'| \geq k|R|$ & $G[R']$ ✂ S connected

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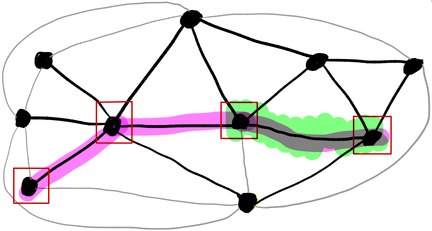

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(2)

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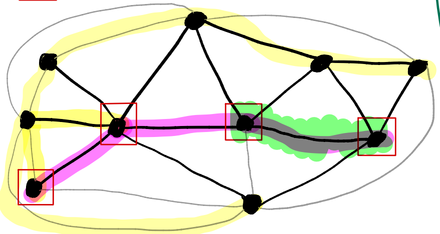

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2

$\boxed{\exists T \text{ spanning tree in } G \setminus R'}$

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③ $\forall r \in R'$

$G \setminus N(r)$ is NOT connected

$\Downarrow$

S ✂ $G[N(r)]$ is NOT connected

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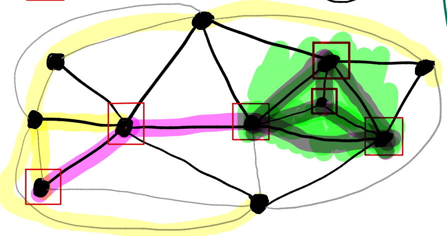

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$\exists T$ spanning tree in $G \setminus R$

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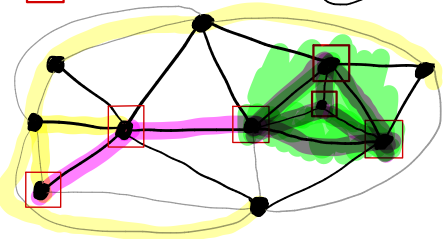

④ $\forall r \in R'$ $\exists u \in N(r)$ st
$u \notin R'$

So we extend T by R'
being leaves

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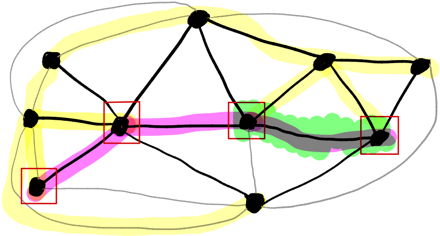

$\square$

Open Problems

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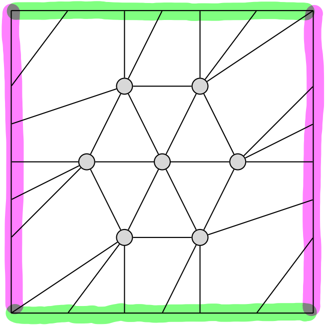
Determine Surface Connectivity for Higher Surfaces

For S **torus**: $\kappa_s(S) \leq \frac{3}{7}$. **Is it also $\kappa_s(S) \geq \frac{3}{7}$?**



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*Is the **maxleaf number** of a planar triangulation on n vertices always **at least $\frac{2}{3}n$**?*

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