Flexibility of Planar Graphs

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Definitions

A weighted request is a function \( w \) that to each pair \((v, c)\) with \( v \in V(G) \) and \( c \in L(v) \) assigns a nonnegative real number.

Let \( w(G, L) = \sum_{v \in V(G), c \in L(v)} w(v, c) \). For \( \varepsilon > 0 \), we say that \( w \) is \( \varepsilon \)-satisfiable if there exists an \( L \)-coloring \( \varphi \) of \( G \) such that \( \sum_{v \in V(G)} w(v, \varphi(v)) \geq \varepsilon \cdot w(G, L) \).

We say that a graph \( G \) with the list assignment \( L \) is:

- \( \varepsilon \)-flexible if every request is \( \varepsilon \)-satisfiable,
- weighted \( \varepsilon \)-flexible if every weighted request is \( \varepsilon \)-satisfiable.

Previous Results

Dvořák, Norin, Postle: List coloring with requests. JGT 19'

There exists \( \varepsilon > 0 \) such that every planar graph

- of girth at least 12 and with an assignment of lists of size 3 is \( \varepsilon \)-flexible,
- of girth at least 5 and with an assignment of lists of size 4 is \( \varepsilon \)-flexible.

with an assignment of lists of size 6 is \( \varepsilon \)-flexible.

Our Results

There exists \( \varepsilon > 0 \) such that every planar graph

- of girth at least 6 and with an assignment of lists of size 3 is weighted \( \varepsilon \)-flexible,
- without triangles and with an assignment of lists of size 4 is \( \varepsilon \)-flexible,
- without 4-cycles and with an assignment of lists of size 5 is weighted \( \varepsilon \)-flexible.

Key Technique

For a function \( f : V(G) \rightarrow \mathbb{Z} \) and a vertex \( v \in V(H) \), let \( f_v \) denote the function such that \( (f_v)(w) = f(w) \) for \( w \neq v \) and \( (f_v)(v) = 1 \).

Suppose \( H \) is an induced subgraph of another graph \( G \). For integers \( k \geq 3 \) and \( d \geq 0 \), we say that \( H \) is a \( (d, k) \)-reducible induced subgraph of \( G \) if

\begin{align*}
\text{FIX} & \quad \text{for every } v \in V(H), H \text{ is } L\text{-colorable for every } (k + \deg_G - \deg_H \downarrow v)\text{-assignment } L, \text{ and}
\text{FORB} & \quad \text{for every } d\text{-independent set } I \text{ in } H \text{ of size at most } k - 2, H \text{ is } L\text{-colorable for every } (k + \deg_G - \deg_H - 1)\text{-assignment } L.
\end{align*}

Lemma. For all integers \( g, k \geq 3 \) and \( b \geq 1 \), there exists \( \varepsilon > 0 \) as follows. Let \( G \) be a graph of girth at least \( g \). If for every \( Z \subseteq V(G) \), the graph \( G[Z] \) contains an induced \( (g - 3, k) \)-reducible subgraph with at most \( k \) vertices, then \( G \) with any assignment of lists of size at least \( k \) is weighted \( \varepsilon \)-flexible.

Reducible Configurations

Without 4-cycle

Discharging . . .

Triangle-free

Open Problems

Is there \( \varepsilon > 0 \) such that every planar graph

- with an assignment of lists of size 5 is (weighted) \( \varepsilon \)-flexible?
- of girth at least 5 and with an assignment of lists of size 3 is (weighted) \( \varepsilon \)-flexible?
- without 4-cycles and with an assignment of lists of size 4 is (weighted) \( \varepsilon \)-flexible?