Parameterized Complexity of Fair Vertex Evaluation Problems

Dušan Knop, Tomáš Masařík, Tomáš Toufar

Algorithmics and Computational Complexity Group, TU Berlin

Graph Problems

Graph Problems are usually tightly connected to a property π

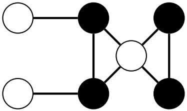
Decision Graph Problem

Input $G = (V, E)$ and a positive integer k.

Question Is there a set of vertices $U \subseteq V$ of size at most k such that U fulfills π in G?

Examples of π

- U intersects every edge of G — vertex cover
- $G \setminus U$ is a forest — feedback vertex set
- ...



Fair Graph Problems

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Measuring locality of U in $G = (V, E)$

We measure how many vertices of U are neighboring a vertex v

$$|N(v) \cap U| \ .$$

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Question Is there a set of vertices $U \subseteq V$ such that

- U fulfills π in G and
- $|N(v) \cap U| \leq k$ for every vertex $v \in V$? $\quad \text{fair}_G(U) \leq k$

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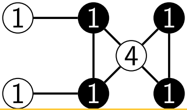

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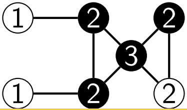

Previous Results—Structural Parameters

Previous Results—graph models with fair objective

- $G, U \models \varphi(U)$
- $\max_{v \in V} |N(v) \cap U| \leq k$ $\quad$ $\mathrm{fair}_G(U) \leq k$

$\rightarrow$ Fair-FO-Evaluation if φ is a first order formula

Theorem (Kolman, Lidický, & Sereni '10)

Fair-MSO$_1$-Evaluation *is in XP parameterized by treewidth of* G *and* $\|\varphi\|$*, size of the formula* φ.

Theorem (Masařík & Toufar '15)

Fair-MSO$_1$-Evaluation *is*

- *in FPT parameterized by neighborhood diversity of* G *and* $\|\varphi\|$ *and*
- *W[1]-hard parameterized by treedepth and feedback vertex number of* G. $\quad$ $\exists\varphi_{\mathrm{hard}}$

Our Results

- propose study of specific problems—Fair Vertex Cover

Theorem

Fair Vertex Cover *is*

- *in FPT parameterized by modular-width of G and*
- *W[1]-hard parameterized by treedepth and feedback vertex number of G.*

Our Results

- propose study of specific problems—FAIR VERTEX COVER

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FAIR VERTEX COVER *is*

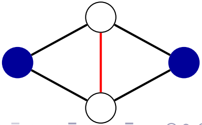
- *in FPT parameterized by modular-width of* G *and*
- *W[1]-hard parameterized by treedepth and feedback vertex number of* G*.*

Theorem

FAIR-MSO$_1$-EVALUATION *is in FPT parameterized by twin-cover number of* G *and* $\|\varphi\|$*.*

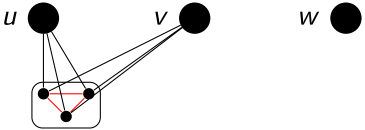
A set of vertices U is a twin-cover of G if each edge is

- covered by U or
- a twin edge.



Twin-cover and the Structure of a Graph

- the cover set X
- a collection of twin-cliques—disjoint cliques whose vertices have the same closed neighborhood



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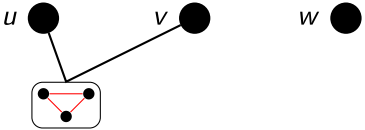

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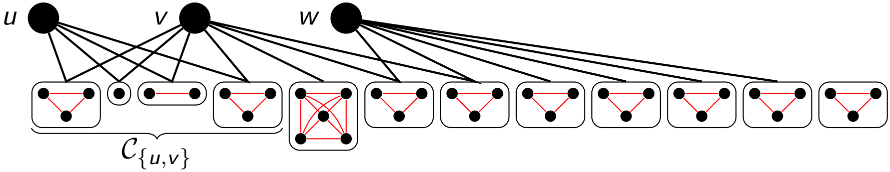

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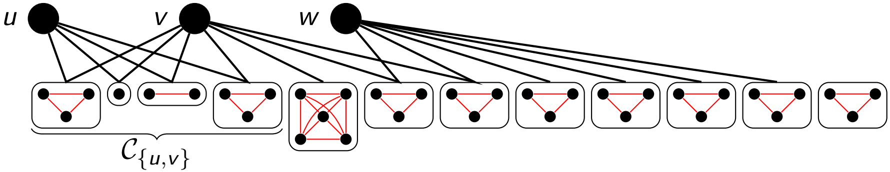

Observation

It suffices to measure the fair cost on twin-cover vertices $\mathcal{C}_\emptyset$.

→ split G into $\mathcal{C}_A$ for $A \subseteq X$

→ minimize independently $|U \cap \mathcal{C}_A|$ for $A \subseteq X$

High-level Overview of the Algorithm

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Twin-cover and the Structure of a Graph

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1. **Reducing the input graph**
   - reduce G to G' such that $G \models \varphi$ if and only if $G' \models \varphi$ and $|G'|$ is bounded by $g(\mathrm{tc}(G), \|\varphi\|)$
   - from G' deduce possible solution shapes (of twin-cliques) in G for φ

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   - realizes the shape (obeys prescribed structural properties)
   - minimizes $\max_{v \in V} |N(v) \cap U|$ suffices for $v \in X$
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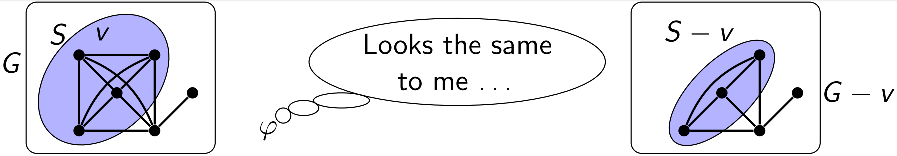
   - IP in fixed dimension with minimization of separable convex objective
3. **Return the best realizer**

Reducing the Input Graph

Lemma (Lampis '12)

Let φ be an MSO_1 formula and let G be a graph. If there is a set S of more than $f(\|\varphi\|)$ vertices having the same closed neighborhood, then for any $v \in S$ we have $G \models \varphi$ if and only if $G - v \models \varphi$.



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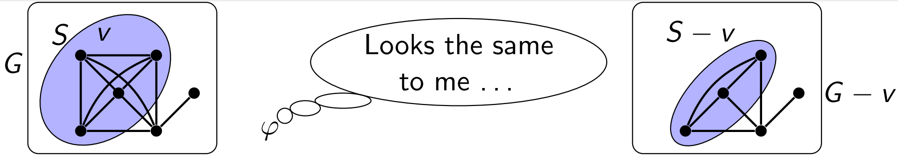

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An MSO_1 formula φ cannot "distinguish more than $f(\|\varphi\|)$ vertices in a twin-clique".

→ directly allows to reduce size of twin-cliques
→ indirectly allows to reduce the number of twin-cliques

} yields G'

Shape

Corollary (informal)

An MSO_1 formula φ cannot "distinguish more than $f(\|\varphi\|)$ vertices in a twin-clique". *in or out of U*

Let $\mathcal{C}_A$ be the collection of twin-cliques C such that $N(v) \cap X = A$ for a vertex $v \in C$ or each $A \subseteq X$.
Given $x, y \in \{0, 1, \ldots, f(\|\varphi\|)\}$ define

A-shape—$\mathcal{S}_A(x, y)$

Specifies the number of twin-cliques $C \in \mathcal{C}_A$ with

- (at least) x vertices in $C \cap U$,
- (at least) y vertices in $C \setminus U$, and
- size exactly/at least $x + y$.

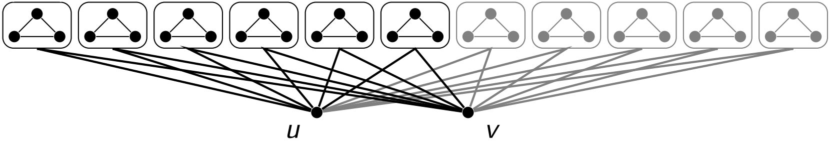
$f(\|\varphi\|)$ yields **at least**

A-Shape Realization Small Cliques

- small cliques in terms of size $|C| \le f(\|\varphi\|)$ no reduction of size done
- some cliques (in $\mathcal{C}_A$ with $|C| = x + y$) were possibly removed in G'
- the shape **exactly** determines both $C \cap U$ and $C \setminus U$ (by x and y)
- specifies (at least) how many cliques of such (x, y)-kind
- choose greedily (x, y)-kind to fill up

Example: $A = \{u, v\}$

- in total 11 cliques of size 3 whose neighborhood in C is exactly $\{u, v\}$



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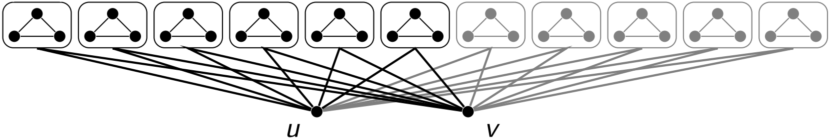

(0, 3)	(1, 2)	(2, 1)	(3, 0)	
2	2	1	1	realizable in G' (but not in G)
2	≥ 3	1	≥ 3	realizable in G (many different)

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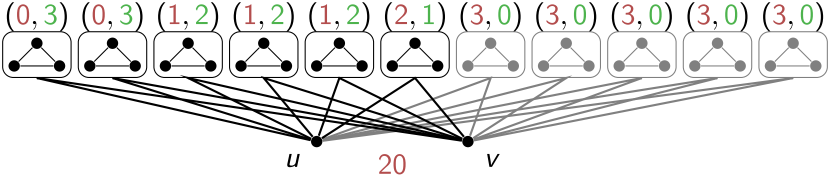

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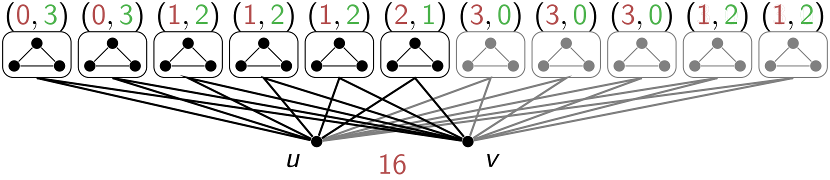

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- FAIR-MSO_1-EVALUATION is in FPT parameterized either by neighborhood diversity or by twin-cover number:
  - Is FAIR-MSO_1-EVALUATION in FPT parameterized by modular-width?
  - Is ANTI-FAIR-MSO_1-EVALUATION in FPT parameterized by twin-cover number?
    $|N(v) \cap U| \leq k \quad \forall v \in V \quad \rightarrow \quad |N(v) \cap U| \geq k \quad \forall v \in V$
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Thank you!