Constant Congestion Brambles in Directed Graphs

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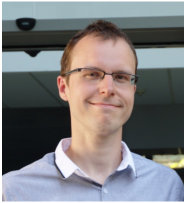

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Treewidth & Grid

Theorem (Undirected grid by Robertson & Seymour '86)

For every $k \geq 1$ *there exists* $t = f(k)$ *such that every graph of treewidth at least* t *contains a* $k \times k$ *grid as a minor.*

Theorem (Undirected grid by Chuzhoy & Tan '21)

For every $k \geq 1$ *there exists* $t = \mathcal{O}(k^9 \mathrm{polylog}(k))$ *such that every graph of treewidth at least* t *contains a* $k \times k$ *grid as a minor.*

Theorem (Directed grid Kawarabayashi & Kreutzer '15)

For every $k \geq 1$ *there exists* $t = f(k)$ *such that every* ***directed*** *graph of* ***directed*** *treewidth at least* t *contains a* $k \times k$ ***directed*** *grid as a minor.*

"Relaxed Grid" — Bramble

Definition

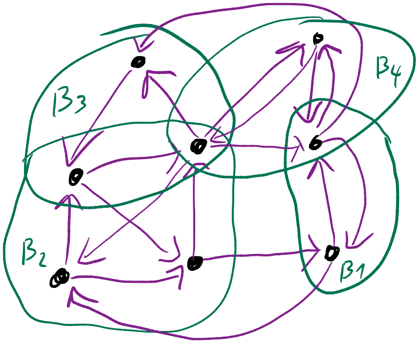
B_1, B_2, B_3, B_4

A **directed bramble** is a family of strongly connected subgraphs s.t.:

- every two subgraphs either **intersect in a vertex**, or
- the graph contains an **arc from one to the other** and an **arc back**.

- **Order**: **min** size of a vertex set that intersects every element of a bramble. 2
- **Size**: the number of its elements. 4
- **Congestion**: **max** number of elements that contain a single vertex. 3

For any bramble:
size $\leq$ order $\cdot$ congestion.



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Theorem (MPRzS ’21)

For every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of ***congestion at most*** 8 *and size at least k.*

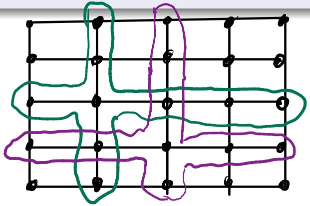
Brambles — Undirected Graphs

For any bramble: size $\leq$ order $\cdot$ congestion.

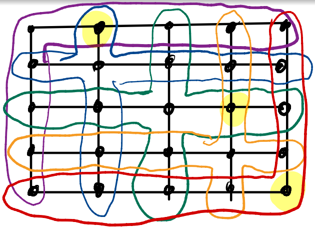
$k \times k$ grid contains a bramble of **order** k and **size** k^2, but **congestion** $2k-1$.

5×5 grid

$\forall_{ij}$ row i
Column j



$k \times k$ grid contains a bramble of **congetion** 2, **order** $\lceil k/2 \rceil$ and **size** k.



Order 3

History — Undirected Graphs

Theorem (Seymour, Thomas '93)

Max order of a bramble is ***exactly*** *its treewidth + 1.*

Theorem (Grohe, Marx '09 and Hatzel, Komosa, Pilipczuk, Sorge '20)

There are classes of graphs where for each $0 < \delta < 1/2$ any bramble of order $\tilde{\Omega}(k^{(0.5+\delta)})$ requires ***exponential*** *size in $k^{2\delta}$.*

For any bramble: size $\leq$ order $\cdot$ congestion.

History — Directed Graphs

Theorem (Reed '99)

Max order of a bramble is ***up to constant factor*** *its treewidth.*

Theorem (Grohe, Marx '09)

There are classes of graphs where for each $0 < \delta < 1/2$ any bramble of order $\tilde{\Omega}(k^{(0.5+\delta)})$ requires ***exponential*** *size in $k^{2\delta}$.*

For any bramble: size $\leq$ order $\cdot$ congestion.

Theorem (MPRzS '21)

For every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of ***congestion at most*** *8 and size at least k.*

History — Directed Graphs

Theorem (MPRzS '21)

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Directed grid give **congestion** 2 bramble of linear size.

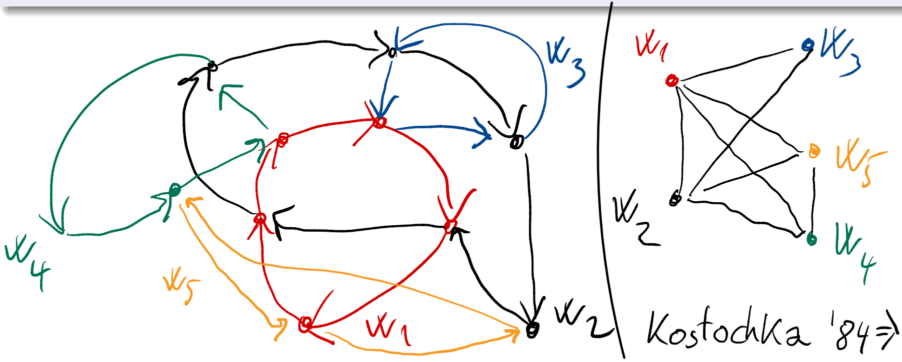
Hence,

- Kawarabayashi and Kreutzer '15 gives congestion 2 but **small-sized** bramble.
- Half-integral grid (Kawarabayashi, Kobayashi, Kreutzer '14) gives congestion 4 but **small-sized** bramble.
- **Planar graphs** grid (Hatzel, Kawarabayashi, Kreutzer '19) gives congestion 2 polynomial bound on a bramble.

Proof — Extracting a bramble (Dense case)

Lemma (Dense winning scenario)

Let c_{KT} be the constant from Kostochka '84. If a graph G contains a family $\mathcal{W}$ of ***closed walks*** *of congestion α, whose intersection graph is* ***not*** *$c_{KT} \cdot d \cdot \sqrt{\log d}$-degenerate, then G* ***contains a bramble of congestion*** *α* ***and size*** *d.*



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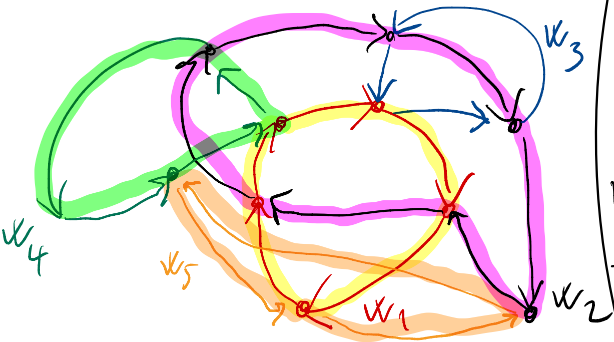
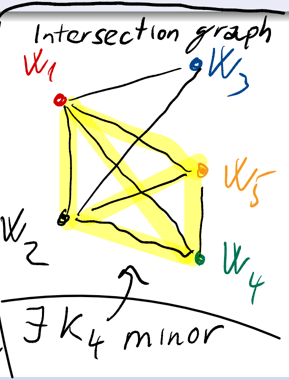




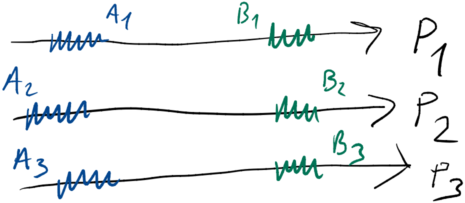
Proof — Starting Point

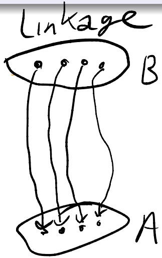
Definition (Path system)

Let $a, b \in \mathbb{N}$. An (a,b)-*path system* $(P_i, A_i, B_i)_{i=1}^{a}$ consists of

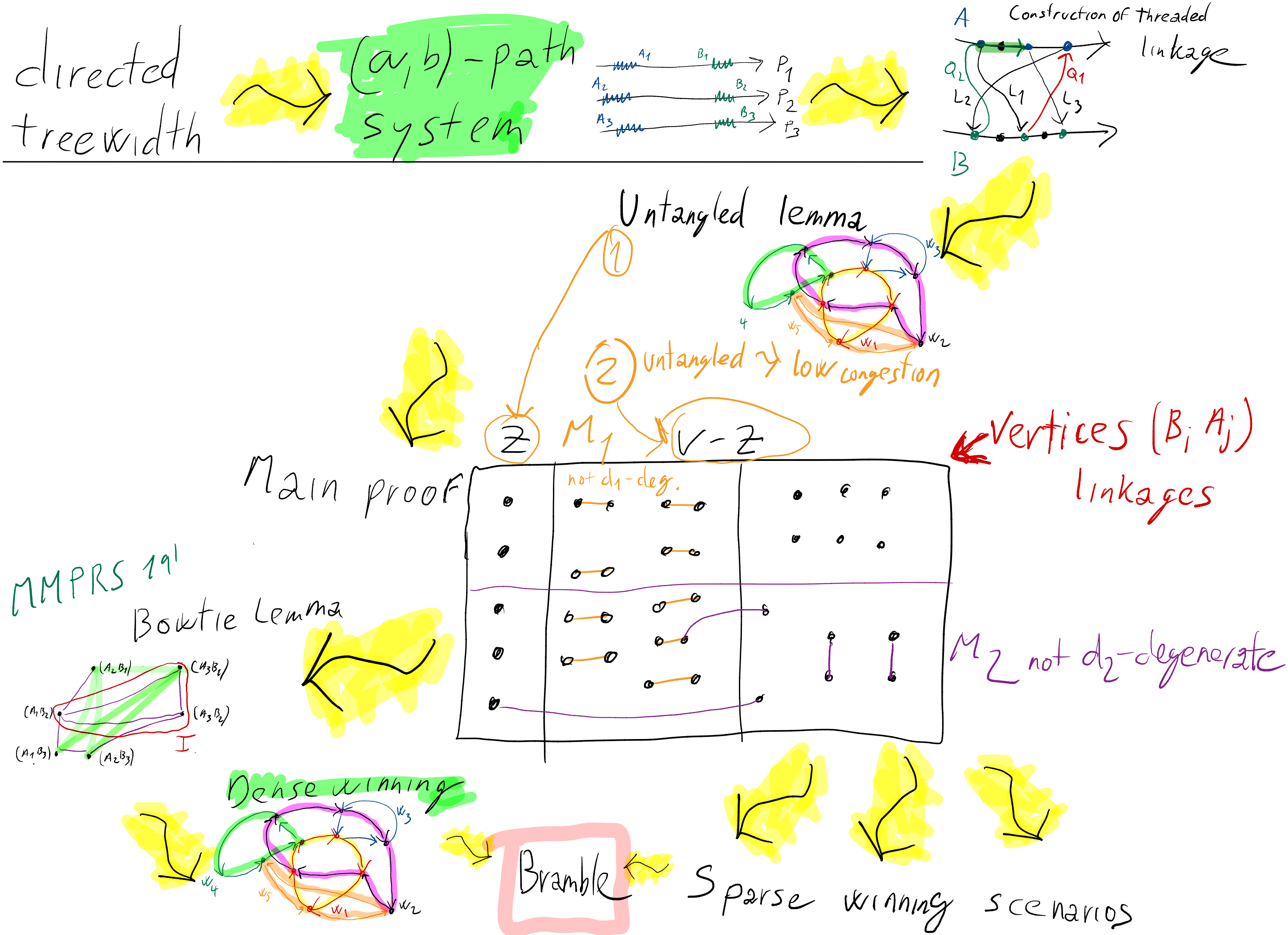
- vertex-disjoint paths $P_1, P_2, \ldots, P_a$, and
- for every $i \in [a]$, two sets $A_i, B_i \subseteq V(P_i)$, each of size b, such that every vertex of B_i appears on P_i later than all vertices of A_i,

such that $\bigcup_{i=1}^{a} A_i \cup B_i$ is **well-linked** in G.





Kawarabayashi, Kreutzer '15:
G **has directed treewidth** $ca^2b^2 \Rightarrow G$ **contains** (a,b)**-path system.**


directed treewidth
(a,b)-path system
Construction of threaded linkage
A
B
Untangled lemma
1
2 untangled → low congestion
Z
M1
V - Z
Main proof
not d1-deg.
Vertices (Bi Aj) linkages
M2 not d2-degenerate
MMPRS 19'
Bowtie Lemma
Dense winning
Bramble
Sparse winning scenarios