Tuza's Conjecture for Threshold Graphs

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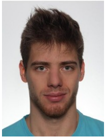




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Easy: $\tau(G) \leq \mathbf{3}\mu(G)$ True

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- confirmed in some graph classes:
  - planar graphs, [Tuza '90]
  - cliques, [Feder, Subi '12]
  - graphs of treewidth at most 6, [Botler, Fernandes, and Gutiérrez '21]
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Only confirmed for a few hereditary classes.

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Confirmed in no superclass of cliques.

Our Results: Threshold Graphs

- interesting herediary classes & superclasses of cliques:

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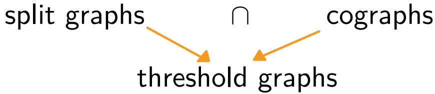
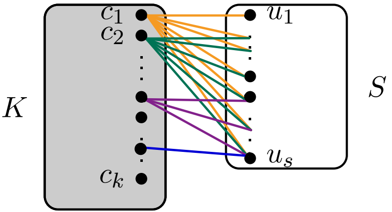


$G = (V,E)$ is a **threshold graph** if its vertices can be partitioned into a clique $K = \{c_1, \ldots, c_k\}$ and an independent set $S = \{u_1, \ldots, u_s\}$:

- $N[c_{i+1}] \subseteq N[c_i]$ for all $1 \leq i < k$ and
- $N(u_i) \subseteq N(u_{i+1})$ for all $1 \leq i < s$.

nested neighborhood



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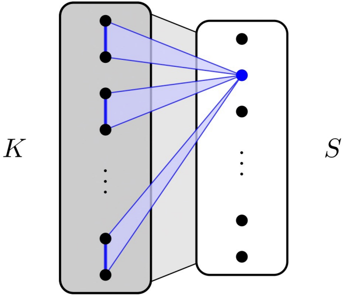
Our result G threshold graph: $\tau(G) \leq 2\mu(G)$

Tools

K clique, S independent set such that they have all the edges in between

1. If $|S| < |K|$, then we can find a triangle packing of size $|S| \cdot \lfloor \frac{1}{2}|K| \rfloor$.
2. If $|S| \geq |K|$, then we can find a triangle packing of size $\binom{|K|}{2}$.



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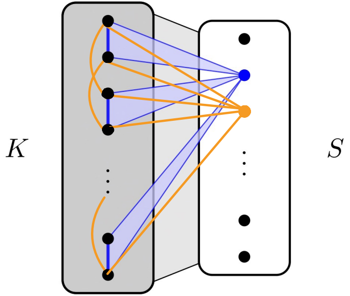

Proof Sketch: Initial Settings

For every threshold graph $G = (V,E)$ there exists a threshold representation (K,S) with a vertex $v \in K$, $N(v) \cap S = \emptyset$.

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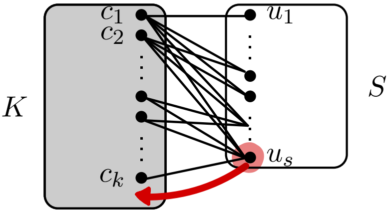

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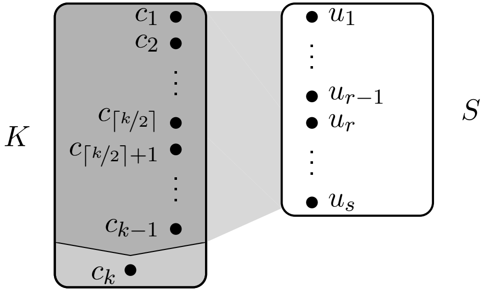

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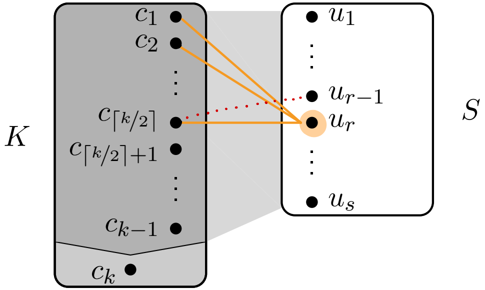

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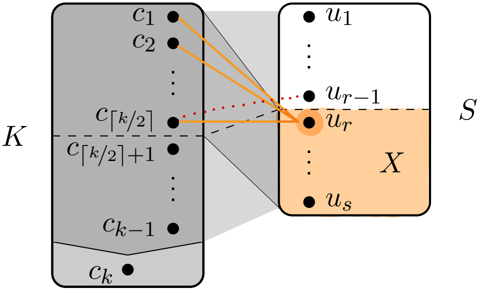

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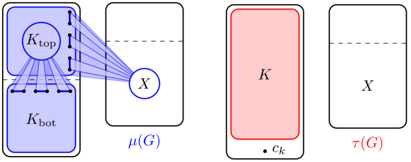
k even

$|X| \geq {}^{k}/_{2}$



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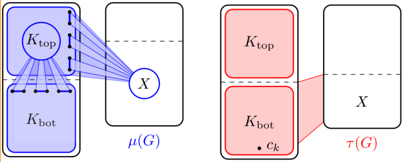

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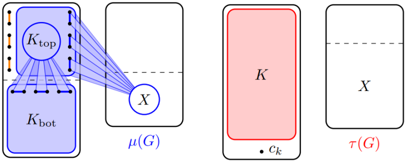

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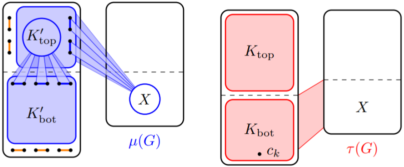

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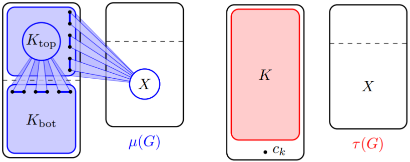

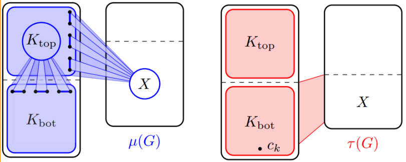

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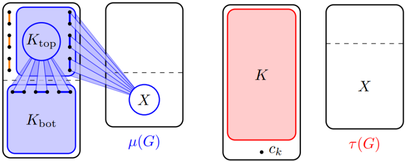
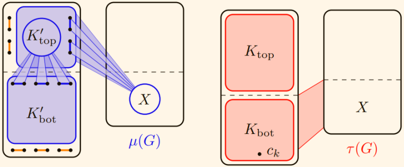
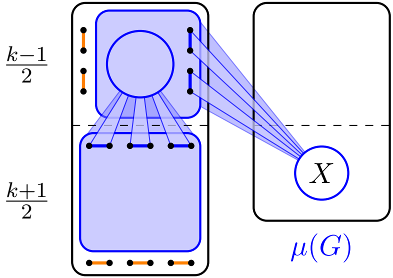




Proof Sketch: Case k odd and $|X| < \frac{k+1}{2}$



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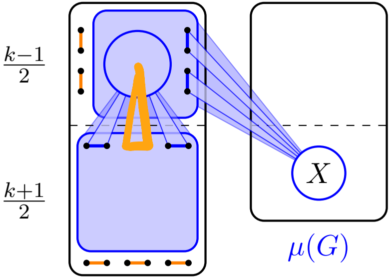

- $\left\lfloor \frac{(k+1)/2}{2} \right\rfloor \cdot \frac{k-1}{2}$

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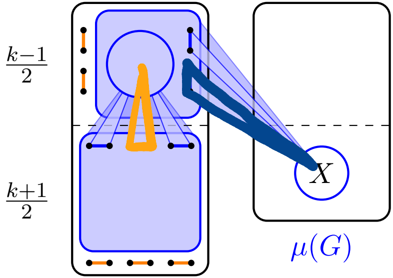

- $\lfloor \frac{(k+1)/2}{2} \rfloor \cdot \frac{k-1}{2}$
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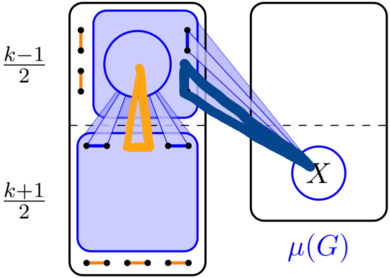

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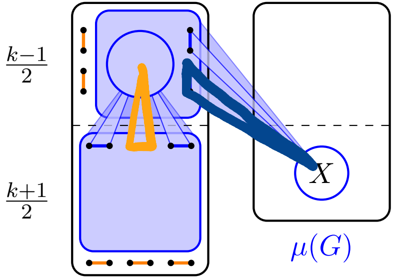

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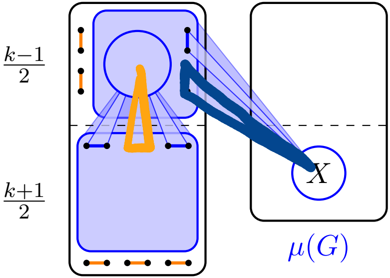

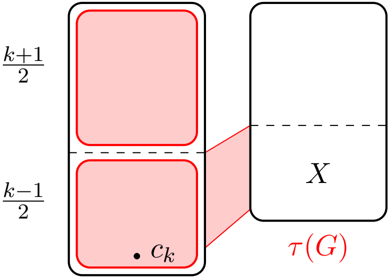

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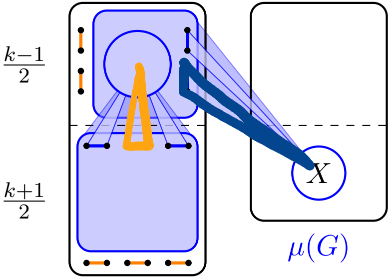

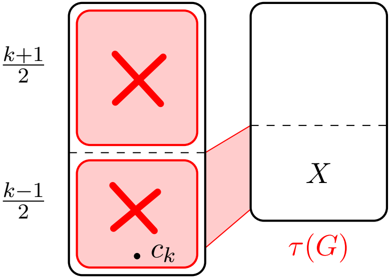

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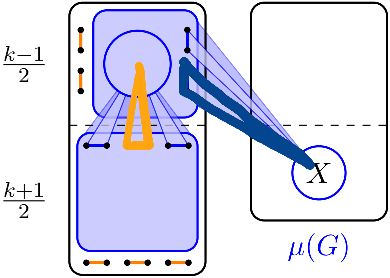

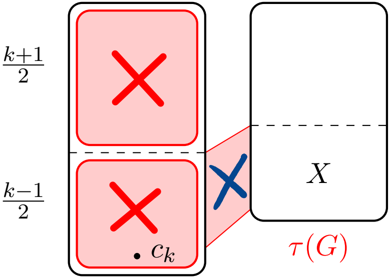

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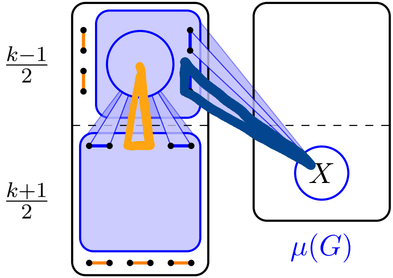

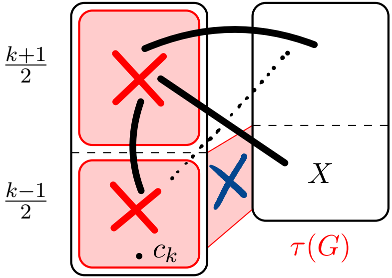

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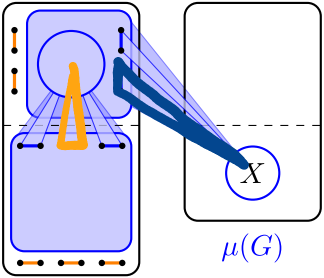
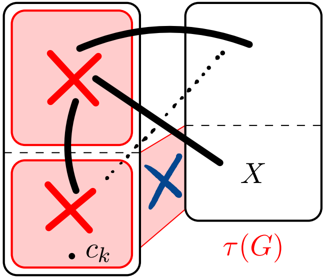
$$\binom{(k+1)/2}{2} + \binom{(k-1)/2}{2} + |X| \frac{k-3}{2} \geq \tau(G)$$

Proof Sketch: Case k odd and $|X| < \frac{k+1}{2}$





- $\lfloor \frac{(k+1)/2}{2} \rfloor \cdot \frac{k-1}{2} \geq \frac{k-1}{4} \cdot \frac{k-1}{2}$
- $\min\{|X| \cdot \lfloor \frac{k-1}{4} \rfloor, \binom{(k-1)/2}{2}\}$
  $\geq |X| \frac{k-3}{4}$

- $\binom{(k+1)/2}{2} + \binom{(k-1)/2}{2}$
- $|X| \frac{k-3}{2}$

$$\mu(G) \geq \frac{k-1}{2} \cdot \frac{k-1}{4} + |X| \frac{k-3}{4}$$

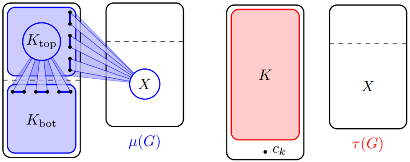
$$\binom{(k+1)/2}{2} + \binom{(k-1)/2}{2} + |X| \frac{k-3}{2} \geq \tau(G)$$

$$2\mu(G) \geq \tau(G)$$

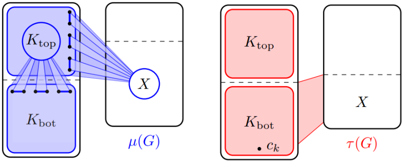
Proof Sketch: Cases

k even

$|X| \geq k/2$

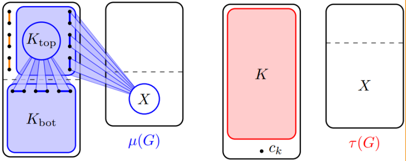


$|X| < k/2$



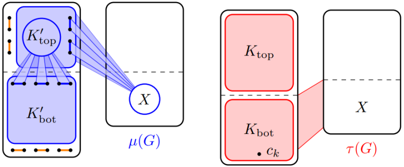
k odd

$|X| \geq k+1/2$



✓

$|X| < k+1/2$



Conclusion

- co-chain graphs

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- co-chain graphs

- interval graphs
- cographs
- split graphs

Conclusion

- co-chain graphs

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Thank you for your attention.