Flexible List Colorings in Graphs with Special Degeneracy Conditions

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Well-established Definitions

List coloring

- A list assignment L for a graph: a function that to each vertex assigns a set of available colors.
- An L-coloring φ of a graph: proper coloring that to each vertex v assigns a color in $L(v)$.

Choosability

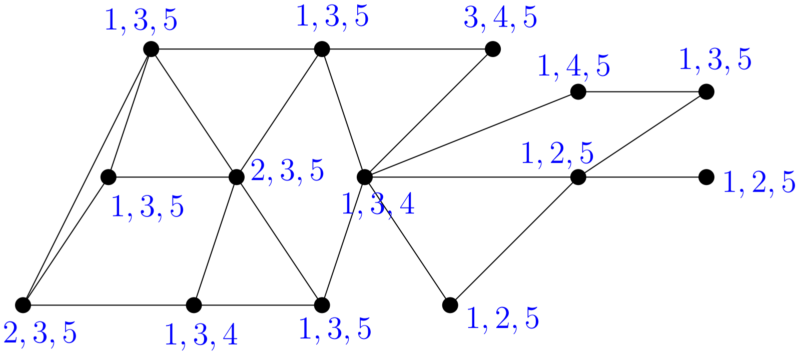
A graph is k-choosable if for every list assignment L of size at least k there exist an L-coloring.

Precoloring Extension

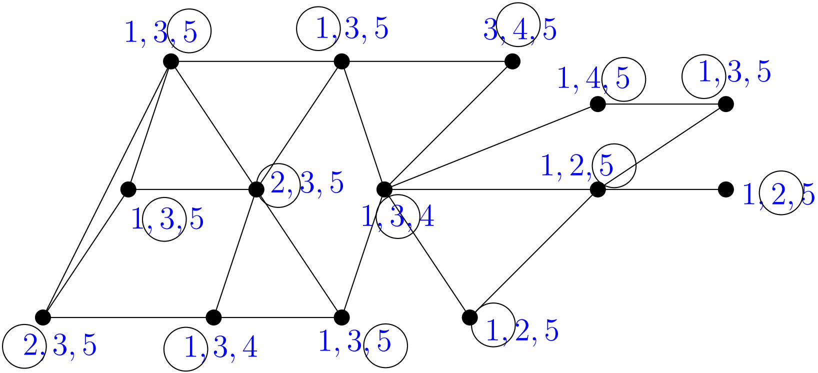
The precoloring extension: Is it possible to extend a proper coloring of a subset of vertices into the whole graph?

Flexibility—example

List assignment of size 3



Flexibility—example



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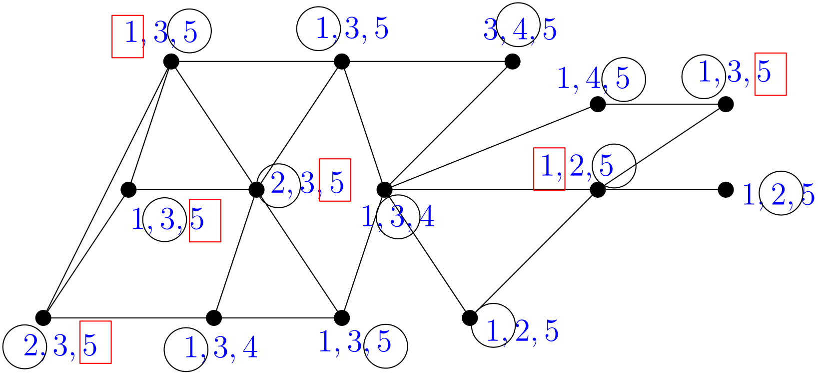

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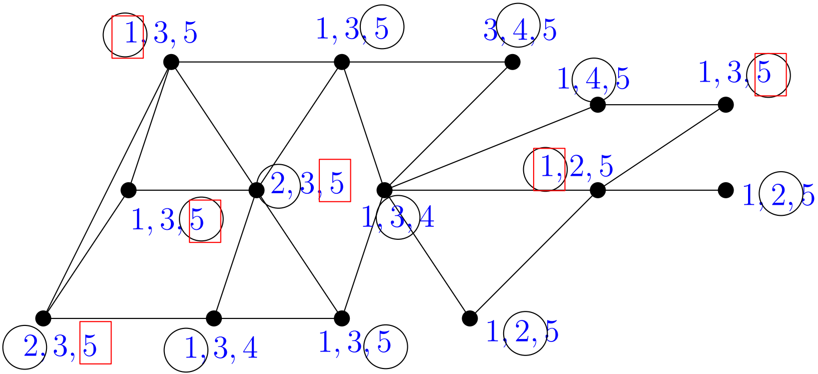


Flexibility

Request

A request on a graph with a list assignment L: a function r that assigns to some vertices a color in $L(v)$.

ε-satisfiable request

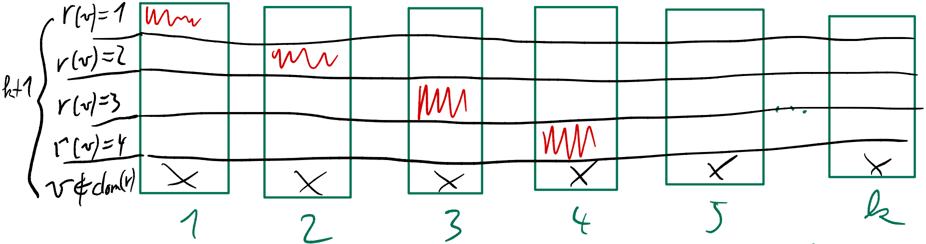
For $\varepsilon > 0$, a request r is ε-satisfiable if there exists an L-coloring ϕ of G such that $\phi(v) = r(v)$ for at least $\varepsilon|\mathrm{dom}(r)|$ vertices $v \in \mathrm{dom}(r)$.

ε-flexibily k-choosability

We say that graph G is ε-flexibily k-choosable if for any list assignment of size at least k, every request is ε-satisfiable.

Lists are necessary!

Any k-colorable graph with any precoloring of size k is $\frac{1}{k}$-flexibl~~e~~y k-ch



$\frac{1}{k}$ fraction of the request is preserved.

$$\frac{(k-1)!}{k!} = \frac{1}{k}$$

Planar Graphs

- 76' Appel, Haken: 4-colorable
- 94' Thomassen: 5-choosable
- 19' Dvořák, Norin, Postle: ε-flexibly 6-choosable
- 19' Dvořák, Norin, Postle: weighted ε-flexibly 7-choosable

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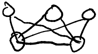
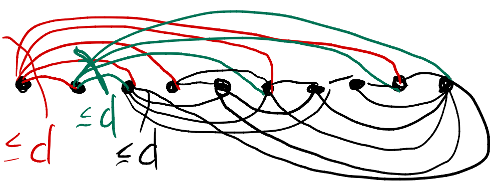
- 20' Dvořák, TM, Musílek, Pangrác: Planar triangle-free wFl 4-Ch
- 20' Dvořák, TM, Musílek, Pangrác: Planar girth 6 wFl 3-Ch
- 19' TM: Planar C_4-free subgraph wFl 5-Ch
- 20+' Choi, Clemen, Ferrara, Horn, Ma, TM: Planar . . . -free
  wFl 5-Ch or wFl 4-Ch
- 20+' Yang, Yang: Planar (C_4, C_5)-free wFl 4-Ch
- 20+' Lidický, TM, Murphy, Zerbib: Planar $(K_4, C_5, C_6, C_7, B_5)$-free
  
  wFl 4-Ch

Degeneracy

Definition (degeneracy)

A graph G is d-degenerate if there is a vertex of degree $\le d$ in every subgraph of G.



degeneracy order

Graphs of degeneracy d

Greedy algorithm gives $(d+1)$-choosability.
Dvořák, Norin, Postle ε-flexibly $(d+2)$-choosability.

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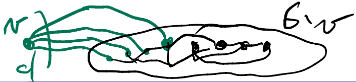
Conjecture (Dvořák, Norin, Postle)

Is it possible to get ε-flexibly $(d+1)$-choosability?
At least for non-regular graphs of max degree $d+1$?

$\exists v \ \deg(v) \leq d$



Degeneracy

Conjecture (Dvořák, Norin, Postle)

Is it possible to get ε-flexibly $(d+1)$-choosability?
Challenging even for $d = 2$.

- 20' Dvořák, TM, Musílek, Pangrác: Planar triangle-free 3-degen. wFl 4-Ch
- 20' Dvořák, TM, Musílek, Pangrác: Planar girth 6 2-degen. wFl 3-Ch
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?	Planar 5-degen 5-ch	Planar girth 5 4degen 4-ch	C_4-free planar 4-degen 4-ch

Our Results

Conjecture (Dvořák, Norin, Postle)

Is it possible to get ε-flexibly $d+1$-choosability? ???

At least for non-regular graphs of max degree Δ? YES!

Theorem (20' BMS)

Graph G of max degree Δ is $\frac{1}{6\Delta}$-flexibly Δ-choosable, unless it is $K_{\Delta+1}$.

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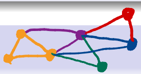


graphs of treewidth 2 (2-trees) are $\frac{1}{3}$-flexibly 3-choosable.

Answering question of Choi, Clemen, Ferrara, Horn, Ma, TM for outerplanar

Theorem (20' BMS)

graphs of treedepth k are $\frac{1}{k}$-flexibly k-choosable.

2-trees

Theorem (20' BMS)

1-Trees are $\frac{1}{2}$-Flex. 2-ch

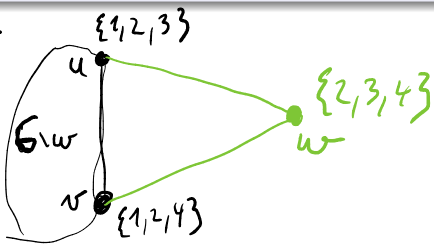
2-trees are $\frac{1}{3}$-flexibly 3-choosable.

Algorithmic: we will construct six colorings:

The properties of the constructed colorings

- each color appears exactly twice at each vertex.
- each pair of colors is unique.

Example:



	u	v	w
1.	1	2	
2.	1	4	
3.	2	1	
4.	2	4	
5.	3	1	
6.	3	2	

2-trees

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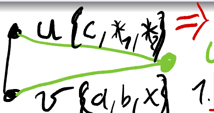
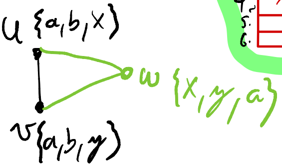
Contradiction min $|L(u) \cup L(v) \cup L(w)|$



1.	x	a
2.	x	b
3.	$*_1$	x
4.	$*_2$	x
5.		
6.		



Unique color c: exchange it with $x \in L(v)$ s.t. $(c, x) \notin \varphi$

$|L(u) \cap L(v)| = 1 \Rightarrow$ unique color

$|L(u) \cap L(v)| = 2$

u	v	w
a	b	
a	y	
b	a	
b	~~y~~	
x	a	
x	b	

unique

Δ-regular graphs

Theorem

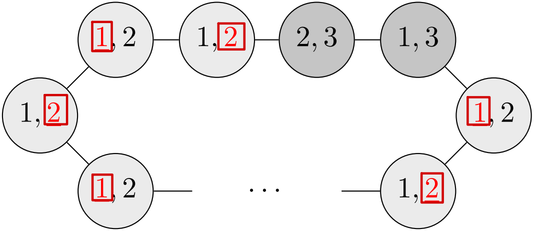
BMS Graph G of max degree $\Delta \geq 3$ is $\frac{1}{6\Delta}$-flexibly Δ-choosable, unless it is $K_{\Delta+1}$

Δ-regular graphs

Theorem

BMS Graph G of max degree $\Delta \geq 3$ is $\boxed{\frac{1}{2\Delta^3}}$-flexibly Δ-choosable, unless it is $K_{\Delta+1}$

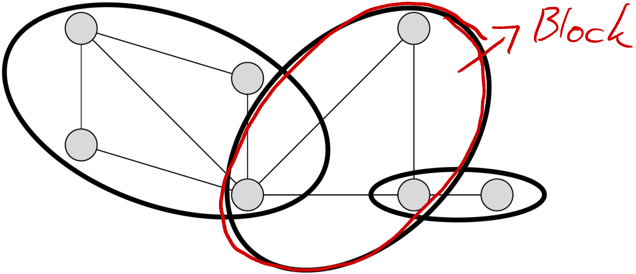
Even cycle is not flexibly 2-choosable!



Block-cut tree and ERT theorem

Definition

A block is a maximal 2-connected subgraph.



Theorem (79' Erdős, Rubin, Taylor)

G is not degree-choosable if and only if every block of G is either a clique or an odd cycle.

Proof: Δ-regular graphs

$\boxed{R \text{ is set of vertices with a request}}$

- Goal: create $R' \subseteq R$ such that $|R'| \geq \varepsilon|R|$ are the satisfied requests.
- Prone R' to be at distance ≥ 4 (keeps at least $\frac{1}{\Delta^3}$).
- Try to list color $G \setminus R'$ with modified lists.
- As $|L(v)| \geq \deg_{G \setminus R'}(v)$ use ERT characterization of degree-choosability.

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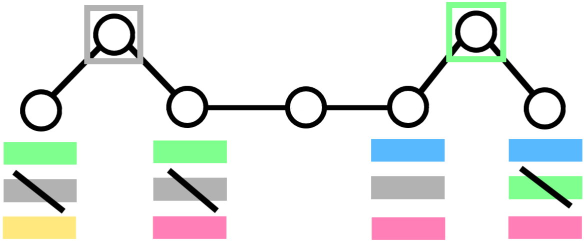

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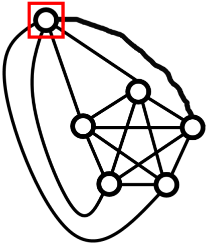
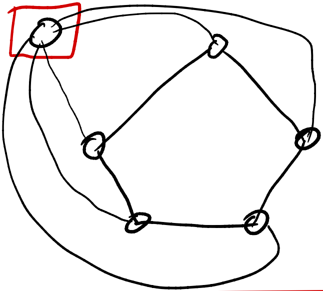
- As $|L(v)| \geq \deg_{G \setminus R'}(v)$ use ERT characterization of degree-choosability.

If all components of $G \setminus R'$ are degree-choosable we are done!

What if some components of $G \setminus R'$ are not deg.-choosable

- If a bad component of $G \setminus R'$ cannot be list-colored by ERT, its terminal blocks must be K_Δ or odd cycle.
- Every bad component has ≥ 2 neighbors in R'.
- Solution: Ignore preferences at $\leq \frac{1}{2}|R'|$ vertices to eliminate bad components.





Recall vertices in R' are in distance ≥ 3 in G.

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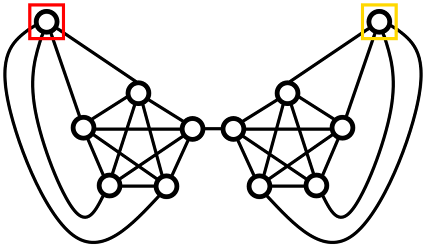
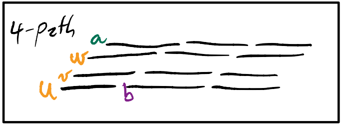


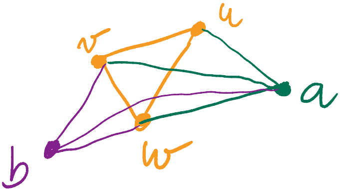
Open Directions

(k-1)-tree

w/

Are k-trees flexibly $(k+1)$-choosable? Even k-path is open, even for $\boxed{k = 4.}$





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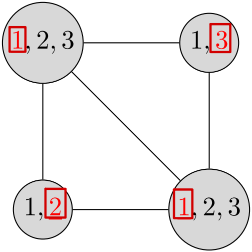
Which graphs are flexibly degree-choosable? e.g.: Diamond is not



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Thank you for your attention!