Parameterized complexity of fair deletion problems II.

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Deletion problems

Given a graph property $P$ and graph $G$, vertex (edge) deletion problem is a task of finding set $S$ of vertices (edges) such that $G \setminus S$ satisfies $P$. 

Examples:

- **Vertex Cover** – $W \subseteq V$ such that $G \setminus W$ has no edges.
- **Feedback Vertex set** – $W \subseteq V$ such that $G \setminus W$ is a forest.
- **Feedback Arc set** – $F \subseteq E$ such that $G \setminus F$ is a DAG.
- **Odd cycle transversal** – $W \subseteq V$ such that $G \setminus W$ is a bipartite.
- **Odd edge cycle transversal** – $F \subseteq E$ such that $G \setminus F$ is a bipartite.

For monotone properties finding any such set is trivial. Usual aim is to find the smallest such set.
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Fair deletion problems

Usually aim is to find smallest set $S$ such that $G \setminus S$ satisfies $P$.

In Fair deletion problems, we want to find set that is “locally” small.

- For a set $F$ of edges, we want to minimize

  $$\max_{v \in V} \deg_F(v).$$

- For a set $W$ of vertices, we want to minimize

  $$\max_{v \in V} |N(v) \cap W|.$$
Parameterized complexity

In parameterized complexity in addition to the input, we have a number called parameter. Examples of parameters:

- size of the solution
- structural parameters (treewidth, clique width, vertex cover...)

Complexity classes

- \( \text{FPT} \) – class of problems solvable in time \( f(k)n^c \)
- \( \text{XP} \) – class of problems solvable in time \( n^{f(k)} \)
- \( \text{W}[1] \)-hard – class of problems that unlikely admit an FPT alg.
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The running time is described as a function of both the size of the input and the parameter.

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Overview of structural parameters
Tree depth

**Definition (Tree depth)**

- The **closure** $Clos(F)$ of a forest $F$ is the graph obtained from $F$ by making every vertex adjacent to all of its ancestors.
- The **tree depth**, denoted as $td(G)$, of a graph $G$ is one more than the minimum height of a rooted forest $F$ such that $G \subseteq Clos(F)$. 
Overview of structural parameters

dense classes

cw
sd
nd
tc

tw

sparse classes

fw
vc

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Neighborhood diversity

Definition (Neighborhood diversity)

The **neighborhood diversity** of a graph $G$ is denoted by $\text{nd}(G)$ and it is the minimum size of a partition of vertices into classes such that all vertices in the same class have the same neighborhood, i.e.

$$N(v) \setminus \{v'\} = N(v') \setminus \{v\},$$

whenever $v, v'$ are in the same class.
Overview of structural parameters

sparse classes

dense classes

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We study properties definable in graph logic ($\text{FO}$, $\text{MSO}_1$, $\text{MSO}_2$).

Sometimes we want to put additional restrictions on the deleted set itself (for example, Connected vertex cover). We use an $\text{MSO}$ formula with one free set variable $S$, such that $G \upharpoonright \phi(S)$ (in contrast to original $G \setminus S \upharpoonright \psi$).

We call that the generalized deletion problem or some authors use monadic second order evaluation.

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Graph properties

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Fair Vertex Cover

Problem formulation

Formulated as a Fair Vertex Deletion problem: Find a set of vertices $W$ such that the rest of the graph has no edges.
Fair Vertex Cover

Problem formulation
Formulated as a Fair Vertex Deletion problem: Find a set of vertices $W$ such that the rest of the graph has no edges.

Our results
- **FPT** algorithm for the Fair vertex cover problem parameterized by modular width.  
  (D. Knop, TM, T. Toufar 2017+)
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Known results

- An **XP** algorithm for the generalized version of the Fair MSO$_2$ edge deletion problem parameterized by treewidth ($f(|\varphi|)n^{O(tw(G))}$). Kolman, Lidický, and Sereni
- Can be easily adapted to the vertex version.
Known results

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- Can be easily adapted to the vertex version.

- MSO$_2$ does **not admit XP algorithm** on cliques unless standard complexity assumption fails by Lampis or Courcelle, Makowsky and Rotics.
Our results — Vertex deletion problem

Positive results

- **FPT** algorithm for generalized version of Fair MSO₁ vertex deletion problem parameterized by *neighbourhood diversity*. (TM, T. Toufar 2017)
- **FPT** algorithm for generalized version of Fair MSO₁ vertex deletion problem parameterized by *twin cover*. (D. Knop, TM, T.Toufar 2017+)
Our results — Vertex deletion problem

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- **FPT** algorithm for generalized version of Fair MSO₁ vertex deletion problem parameterized by **twin cover**. (D. Knop, TM, T. Toufar 2017+)

Hardness results

- The Fair vertex cover problem is **W[1]-hard** parameterized by **tree depth + feedback vertex set**. (D. Knop, TM, T. Toufar 2017+)
Our results — Edge deletion problem

Positive results

- **FPT** algorithm for generalized version of Fair MSO₂ edge deletion problem parameterized by vertex cover. (TM, T. Toufar 2017)
Our results — Edge deletion problem

Positive results

- **FPT** algorithm for generalized version of Fair MSO$_2$ edge deletion problem parameterized by vertex cover.
  (TM, T. Toufar 2017)

Hardness results

- FO Fair deletion is $W[1]$-hard with respect to tree depth + feedback vertex set size.
  (TM, T. Toufar 2017)
Overview of the results

Green means FPT algorithm for \( \text{MSO}_2 \) edge deletion problem.
Orange means FPT algorithm only for \( \text{MSO}_1 \) vertex deletion problem.
Red means hardness results for both edge and vertex deletion problem.
Blue means it is mostly unknown.
Open questions

- Is there a “basic” edge deletion problem such that it is $W[t]$-hard on graphs of bounded tree depth, feedback vertex set or shrub depth?

- Are there other structural parameters where we can obtain FPT algorithms? (e.g. modular width)

- Are there NP-hard fair vertex deletion problems that admit an FPT algorithm parameterized by tree depth (and feedback vertex set)?

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