Maximum Weight Independent Set in Graphs with no Long Claws in Quasi-Polynomial Time

Peter Gartland, Daniel Lokshtanov, Tomáš Masařík,
Marcin Pilipczuk, Michał Pilipczuk, Paweł Rzążewski

MWIS and No Long Claws

Maximum Weighted Independent Set (MWIS)
Input: Graph G with weights on vertices $\mathfrak{w}$ and τ.
Question: Is there a set $S \subseteq V(G)$ such that there are no edges inside S and $\sum_{v \in S} \mathfrak{w}(v) \geq \tau$?

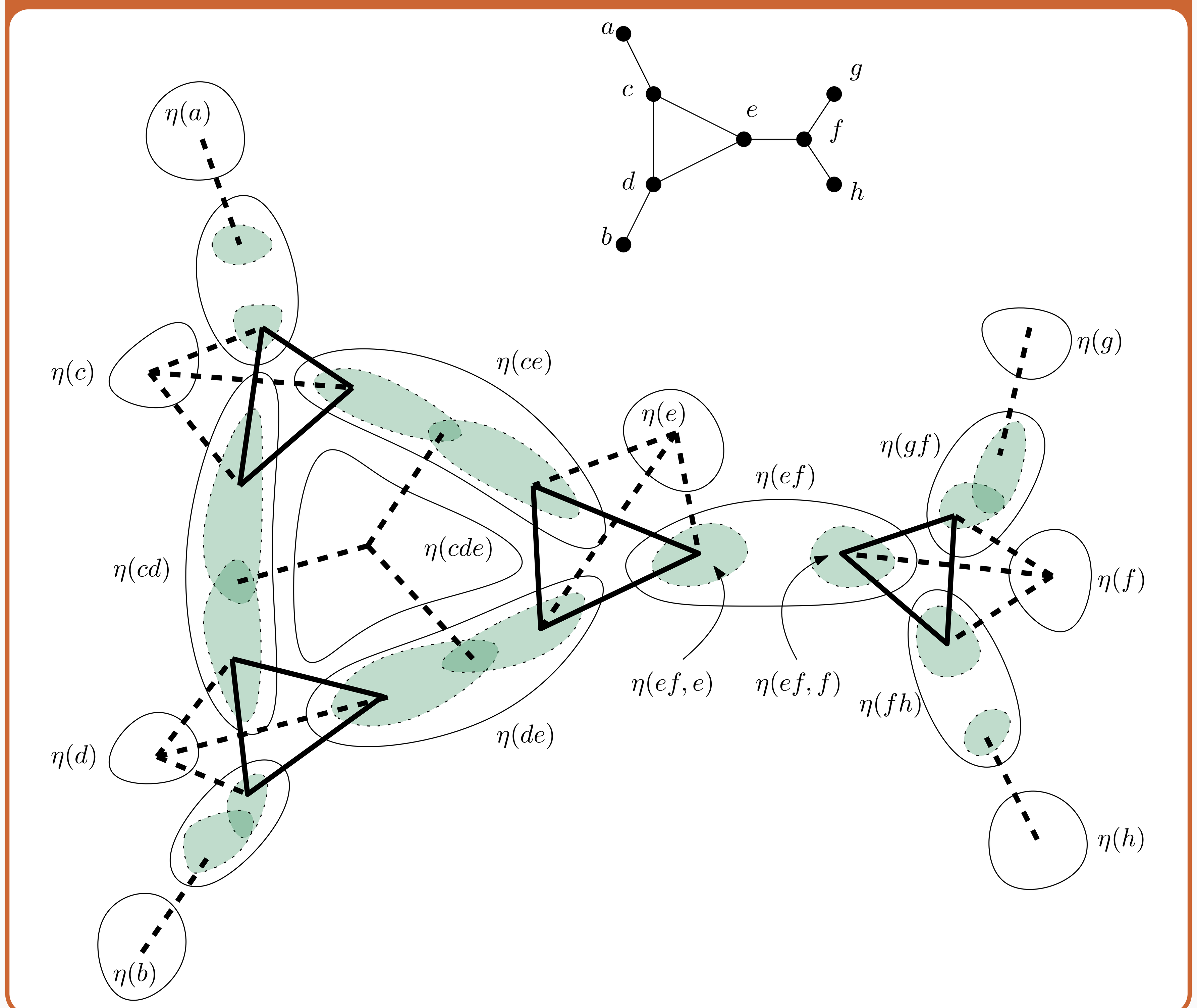
Theorem: MWIS in Quasipolynomial Time [GLMPPR '23]

For every H that is a forest whose every component has at most three leaves, there is an algorithm for the Maximum Weight Independent Set problem in H-free graphs running in time $\mathbf{n}^{\mathcal{O}_H(\log^{19} \mathbf{n})}$.

'19 Grzesik, Klimošová, Pilipczuk, Pilipczuk
$\rightsquigarrow$ Polynomial on P_6-free graphs
'20 Chudnovsky, Pilipczuk, Pilipczuk, Thomassé
$\rightsquigarrow$ QPTAS, subexp. on $S_{t,t,t}$-free graphs
'20 Gartland, Lokshtanov & '21 Pilipczuk, Pilipczuk, Rzążewski
$\rightsquigarrow$ Quasipolynomial on P_t-free graphs
'21 Gartland, Lokshtanov, Pilipczuk, Pilipczuk, Rzążewski
$\rightsquigarrow$ Quasipolynomial on $C_{\geq t}$-free graphs
'22 Abrishami, Chudnovsky, Dibek, and Rzążewski
$\rightsquigarrow$ Polynomial on $S_{t,t,t}$-free graphs of bounded degree

Our Toolbox

Extended strip decomposition



Theorem [Chudnovsky & Seymour '10]: Let G be an n-vertex graph and $Z \subseteq V(G)$ with $|Z| \geq 2$. There is a $\mathcal{O}(n^5)$ algorithm that returns either:
- an **induced subtree** of G containing at least **three elements of** Z, or
- an **extended strip decomposition** (H, η) of (G, Z).

Gyárfás' path analog for $S_{t,t,t}$

Theorem [Majewski, Masařík, Novotná, Okrasa, Pilipczuk, Rzążewski, and Sokołowski '22]: Given an n-vertex graph G and $t \geq 1$, one can in polynomial time either:
- output an induced copy of $S_{t,t,t}$ in G, or
- output a set $\mathcal{P}$ of **at most** $11 \log n + 6$ **induced paths** in G, each of length at most $t+1$, and an **extended strip decomposition** of $G - N[\bigcup_{P \in \mathcal{P}} V(P)]$ whose every particle has weight at most $0.5\mathfrak{w}(G)$, i.e., ***refined***.

k-dominated b-balanced separators

- **DEF:** Set $S \subseteq V(G)$ such that no component of $G - S$ has more than b vertices (or weight) and S is dominated by k vertices.
- Used to show quasipolynomial-time algorithm on P_t-free graphs (**Gartland, Lokshtanov '21**)
- Do not have to exist in $S_{t,t,t}$-free graphs, e.g., line graph of a clique!

Extended Strip Lemma

- Use a tool to get **refined** extended strip decomposition of $G - X$.
- Attempt to **return** vertices in X **one by one** to the extended strip decomposition while keeping it refined.
- If the previous fails we either get **Outcome (i)** or **(ii)**.

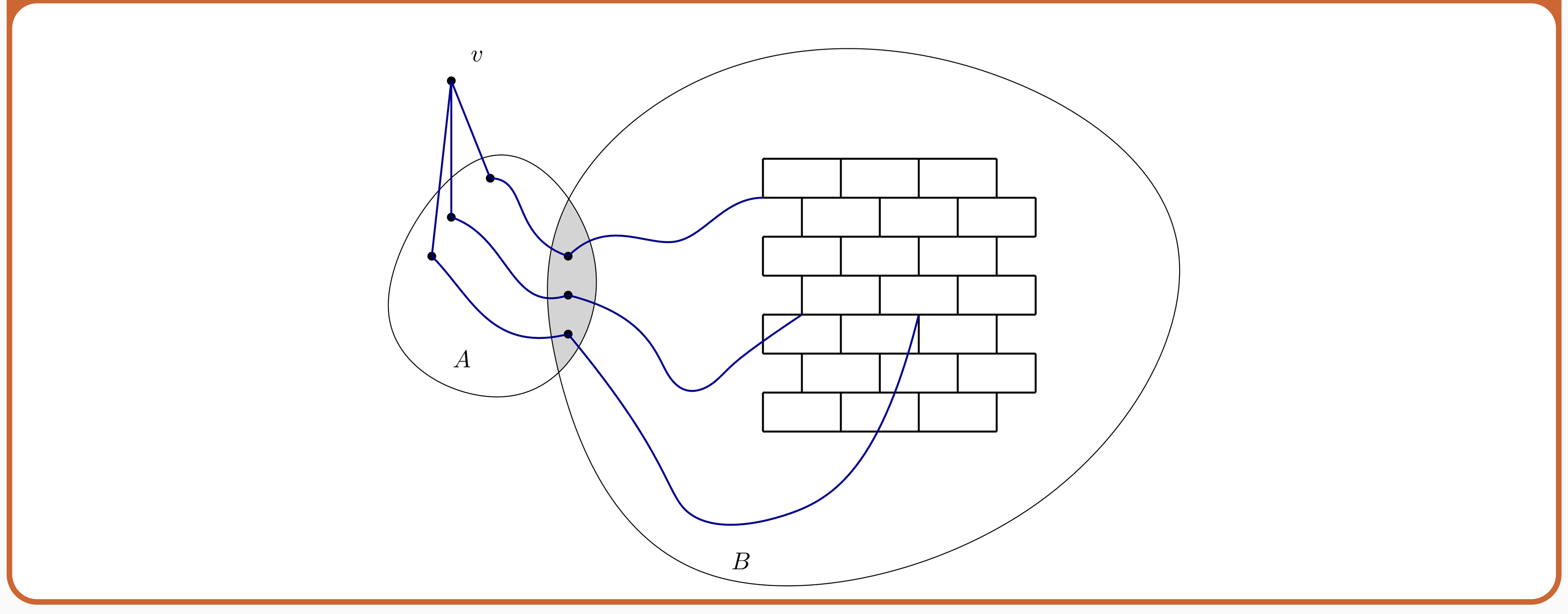
The lemma [GLMPPR '23]: For every fixed integer t there exists an integer c_t and a **polynomial-time algorithm** that, given an n-vertex graph G, a weight function $\mathfrak{w} : V(G) \to [0, +\infty)$, a real $\tau \geq \mathfrak{w}(G)$, a **vertex** $v \in V(G)$, and a **refined extended strip decomposition** (H, η) of $G - v$, returns one of the following:

(i) an induced copy of $S_{t,t,t}$ in G;

(ii) c_t**-dominated** 0.99τ**-balanced separator**;

(iii) a **refined** extended strip decomposition of G.

Extending a subdivided claw to an $S_{t,t,t}$ using the large wall W.



Algorithmic Concepts

Outcome (iii) of the extended strip lemma

Simple divide & conquer strategy on multiplicatively smaller particles.
$\rightsquigarrow$ **Quasipolynomial** branching

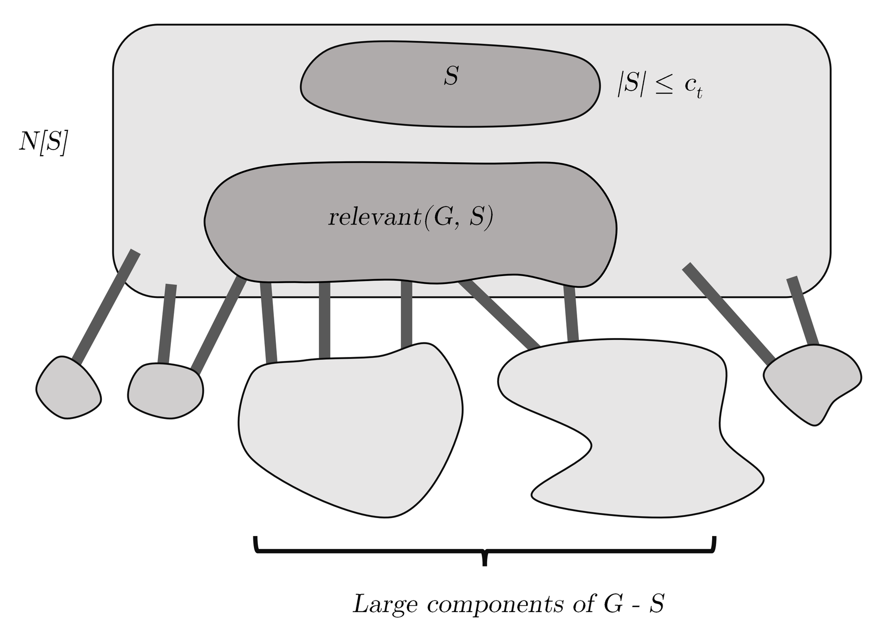
Outcome (ii) of the extended strip lemma

c-boosted balanced separator

Simplified DEF: a set $N[S]$ dominated by a set S of at most c vertices, such that no component of $G - N[S]$ has more than $|V(G)|/16c^2$ vertices.
Packing lemma: Let G be an n-vertex $S_{t,t,t}$-free graph, s an integer, and $\mathcal{F}$ a multi-set of subsets of $V(G)$ such that every set in $\mathcal{F}$ is an s-boosted balanced separator. Assume no vertex belongs to more than c sets of $\mathcal{F}$. Then, provided $|\mathcal{F}| \geq 80sct$, no component of G contains over $3n/4$ vertices.
$\rightsquigarrow$ **Quasipolynomial** branching
- Packing lemma analog is true assuming only k-dominated b-balanced separators in P_t-free graphs,
- but not true for, e.g., path.

Boosting balanced separator

Boosting lemma: Let G be an n-vertex $S_{t,t,t}$-free graph, let $N[S]$ be a **balanced separator** for G dominated by a set S of at most c_t vertices, and let $\mathcal{F}$ be a multi-set of $|\mathsf{relevant}(G,S)|/100c_t^3$**-balanced separators** for $(G, \mathsf{relevant}(G,S))$. Assume no vertex belongs to over c sets of $\mathcal{F}$. If $|\mathcal{F}| \geq 10ct$, either S is a c_t**-boosted balanced separator** or no component of G contains more than $3n/4$ vertices.
$\rightsquigarrow$ **Quasipolynomial** branching



arXiv: 2305.15738