Separator Theorem for Minor-Free Graphs in Linear Time

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IBS Discrete Math Seminar Daejeon 2026

Our Theorem

Balanced Separator

A subset of vertices $S \subseteq V$ is a **balanced separator** of $G = (V, E)$ if **every connected component of $G \setminus S$** has size at most $1/2 \cdot |V|$.

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Alon, Seymour, and Thomas 1990

For any integer $h \geq 1$, any **K_h-minor-free graph** admits a balanced separator of size $O(h^{3/2}\sqrt{n})$ that can be found in time $O(h^{1/2}\sqrt{n}m)$.

Our Theorem

Balanced Separator in Linear Time

Let G be any given graph with n vertices, and $h > 0$ be a parameter. There is an algorithm that runs in deterministic $O(\text{poly}(h)n)$ time and outputs either:

- a balanced separator of size $O(\text{poly}(h)\sqrt{n})$ or
- $\perp$ if G contains a K_h **as a minor**.

Furthermore, in the latter case, we can output a K_h-minor model of G with probability at least $1/2$ in an additional randomized $O(\text{poly}(h)n)$ time.

Comparison of Results

Separator size	Running time	References
$O(h^{3/2}\sqrt{n})$	$O(h^{1/2}\sqrt{n}m)$	Alon, Seymour, Thomas 90
$2^{O(h^2)}n^{2/3}$	$2^{O(h^2)}n$	Reed & Wood 09
$O(h\sqrt{n}+f(h))$	$O(g(h)\,n^{1+\varepsilon})$	Kawarabayashi & Reed 10

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$O(h\sqrt{n} + f(h))$	$O(g(h)\, n^{1+\varepsilon})$	Kawarabayashi & Reed 10
$O(h\sqrt{n \log n})$	$O(h\sqrt{n \log n}m)$	Plotkin, Rao, Smith 94
$O(\mathrm{poly}(h)n^{4/5+\varepsilon})$	$O(\mathrm{poly}(h)\, n)$	Wulff-Nilsen 11
$O(h\sqrt{n \log n})$	$O(\mathrm{poly}(h)\, n^{5/4+\varepsilon})$	Wulff-Nilsen 11
$O(\mathrm{poly}(h)\sqrt{n}\log^2(n))$	$O(\mathrm{poly}(h)\, n\, \mathrm{polylog}(n))$	Räcke, Shah, Täubig 14; Peng 16
$O(\mathrm{poly}(h)\sqrt{n})$	$O(\mathrm{poly}(h)\, n)$	**This paper**

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$O(\text{poly}(h)\sqrt{n})$	$O(\text{poly}(h)\,n)$	**This paper**
$O(\sqrt{n})$	$O(n)$	Lipton & Tarjan 79—**planar graphs**
$O(\sqrt{gn})$	$2^{O(\text{poly}(g))}n$	Gilbert, Hutchinson, Tarjan 84—**genus-g graphs**
		Kawarabayashi, Mohar, and Reed 08
$O(\text{poly}(g)\sqrt{n})$	$O(\text{poly}(g)\,n)$	**This paper**—**genus-g graphs**

Stochastic Connector—Core Technique

Stochastic Connector

A **stochastic connector of size k** for a given graph $G = (V, E)$ is a **set of k trees** $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ such that:

1. $|V(T_i)| \geq n - n/(10k)$ for every $i \in [k]$.
2. For every $i, j \in [k]$ such that $i \neq j$: $\Pr_{P \sim \mathcal{R}(T_i), Q \sim \mathcal{R}(T_j)}[P \cap Q \neq \emptyset] \leq \frac{1}{5k^2}$

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Lemma

Given **stochastic connector** $\mathcal{T}$ of size k in G, for any t such that $20t^2 \leq k$, we can construct a K_t-minor model of G in $O(km)$ time with probability at least $2/5$.

Korhonen Lokstanov 24 Almost-embedding to K_t-minors

For every $h \geq 1$, there **is** a graph H with $|V(H)| + |E(H)| \leq 3h^2$ such that an almost-embedding of H can be turned into a K_h-minor model of G in $O(h^2 m)$ time.

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$\mathcal{P}(G)$ is set of all simple paths in G. An **almost-embedding** of H to G is a $\phi : V(H) \to V(G)$ and $\phi : E(H) \to \mathcal{P}(G)$ such that:

1. for every $e = uv \in E(H)$, $\phi(e)$ is a $\phi(u)$-to-$\phi(v)$ path.
2. for any two edges $e_1 \neq e_2$ of H **that do not share an endpoint**, $\phi(e_1) \cap \phi(e_2) = \emptyset$.

Algorithm—Goal: $(1-\frac{1}{10k})$-balanced separator

```
FindSeparator(G = (V,E)):
    if |E| ⩾ 100h²n                          ⟨⟨G is dense⟩⟩
        return K_h-minor model from Lemma 4
    w_1(v) ← 40 for every v ∈ V
    S ← ∅                                    ⟨⟨The separator⟩⟩
    k ← 20h²
    for t ← 1 to k                           ⟨⟨O(h²) iterations⟩⟩
        (C*_t, S_t, M_t) ← KPR(⟨G, w_t⟩, ⌊√n/6h²⌋, h)     ⟨⟨KPR with Δ = ⌊√n/6h²⌋, applying BFS h times⟩⟩
        if M_t ≠ ∅                           ⟨⟨K_h-minor model found⟩⟩
            return M_t
        S ← S ∪ S_t
        if |C*_t| ⩽ (1 − 1/10k)n             ⟨⟨The current separator is (1 − 1/10k)-balanced⟩⟩
            return S
        c_t ← an arbitrary vertex in C*_t
        T_t ← BFS(⟨G, w_t⟩, c_t, √n)         ⟨⟨BFS truncated at radius √n⟩⟩
        for every v ∈ V(T_t)
            w_{t+1}(v) ← w_t(v) + ⌈(|T_t(v)|k³/√n) · w_t(v)⌉     ⟨⟨Reweighting vertices for next iteration⟩⟩
    return FindMinor(G, {T_1, T_2, ..., T_k})    ⟨⟨Find K_h-minor model by Lemma 3⟩⟩
```

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Vertex-Weighted KPR [Klein, Plotkin, Rao 93 (unweighted)]

Given $\langle G = (V, E), w \rangle$ with integer **weights on vertices**, m edges and integer parameters $\Delta > 0, h > 0$. Let $W = \sum_{v \in V} w(v)$. The procedure $\mathrm{KPR}(\langle G, w \rangle, \Delta, h)$ runs in time $O(h \cdot (m + W))$ and returns either:

- a pair (C^*, S) where S is a subset of vertices of size $h \cdot W/\Delta$, and C^* is the connected component of **maximum size** of $G - S$, which is guaranteed to have (vertex-weighted) **weak diameter** at most $6 \cdot h^2 \Delta$, or
- a K_h-minor model.

Weak diameter is diameter in G (but not in $G - S$).

Weighted BFS

Just weighted BFS, but it runs in linear time!

Algorithm—Goal: $\left(1 - \frac{1}{10k}\right)$-balanced separator

$\underline{\text{FINDSEPARATOR}(G = (V, E))}$:

- if $|E| \geqslant 100h^2 n$ ⟨⟨*G is dense*⟩⟩
  - return K_h-minor model from Lemma 4
- $w_1(v) \leftarrow 40$ for every $v \in V$
- $S \leftarrow \emptyset$ ⟨⟨*The separator*⟩⟩
- $k \leftarrow 20h^2$
- for $t \leftarrow 1$ to k ⟨⟨*$O(h^2)$ iterations*⟩⟩
  - $(C_t^*, S_t, M_t) \leftarrow \text{KPR}(\langle G, w_t\rangle, \lfloor \sqrt{n}/6h^2 \rfloor, h)$ ⟨⟨*KPR with $\Delta = \lfloor \sqrt{n}/6h^2 \rfloor$, applying BFS h times*⟩⟩
  - if $M_t \neq \emptyset$ ⟨⟨*K_h-minor model found*⟩⟩
    - return M_t
  - $S \leftarrow S \cup S_t$
  - if $|C_t^*| \leqslant (1 - \frac{1}{10k})n$ ⟨⟨*The current separator is $(1 - \frac{1}{10k})$-balanced*⟩⟩
    - return S
  - $c_t \leftarrow$ an arbitrary vertex in C_t^*
  - $T_t \leftarrow \text{BFS}(\langle G, w_t\rangle, c_t, \sqrt{n})$ ⟨⟨*BFS truncated at radius $\sqrt{n}$*⟩⟩
  - for every $v \in V(T_t)$
    - $w_{t+1}(v) \leftarrow w_t(v) + \left\lceil \frac{|T_t(v)|k^3}{\sqrt{n}} \cdot w_t(v) \right\rceil$ ⟨⟨*Reweighting vertices for next iteration*⟩⟩
- return $\text{FINDMINOR}(G, \{T_1, T_2, \ldots, T_k\})$ ⟨⟨*Find K_h-minor model by Lemma 3*⟩⟩

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$E_{t,v}$ **be the event** that a random rooted path $P \sim \mathcal{R}(T_t)$ contains v.

Algorithm gives stochastic connector (if does not end earlier)

1. **Invariant 1:** $\Pr[E_{i,v}] \leq (20k^3\sqrt{n})^{-1} w_{t+1}(v)$ for every $i \leq t$ and every v.
2. **Invariant 2:** $\Pr[E_{i,v} \cap E_{j,v}] \leq (10k^6 n)^{-1} w_{t+1}(v)$ for every $i \neq j \leq t$ and every v.
3. **Invariant 3:** $\sum_{v \in V} w_t(v) \leq (40 + (k^3+1)(t-1))n \sim k^4 n$.
4. **Invariant 4:** $|V(T_t)| \geq (1 - \frac{1}{10k})n$.
5. **Invariant 5:** $w_t(v) \geq 40$ for every v.

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1. $|V(T_i)| \geq n - n/(10k)$ for every $i \in [k]$. **Invariant 4**
2. For every $i, j \in [k]$ such that $i \neq j$: $\Pr_{P \sim \mathcal{R}(T_i), Q \sim \mathcal{R}(T_j)}[P \cap Q \neq \emptyset] \leq \frac{1}{5k^2}$ **Inv. 2+3**

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Weight change in step t of the algorithm: $w_{t+1} = \frac{|T_t(v)| k^3 w_t(v)}{\sqrt{n}}$.

Thank you for your attention!