

Computational complexity of distance edge labeling

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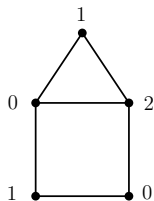
IWOCA 2015

Section 1

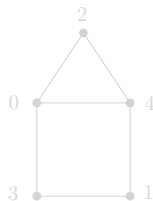
Introduction

Graph coloring

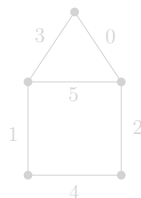
Vertex Coloring



Distance Vertex Labeling



Distance Edge Labeling



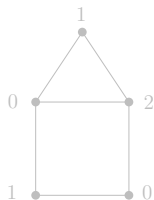
Definition (Vertex-coloring mapping)

Vertex-coloring mapping $f: V \rightarrow \mathbb{N}_0$ satisfies:

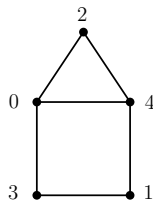
- $|f(v) - f(\tilde{v})| \geq 1$ for **neighboring** vertices v, \tilde{v} .

Graph coloring

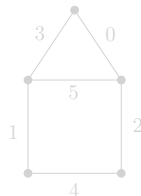
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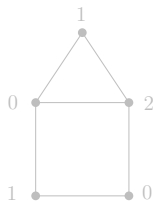
Definition (Vertex-labeling mapping)

Vertex-labeling mapping $f_{(2,1)}: V \rightarrow \mathbb{N}_0$ satisfies:

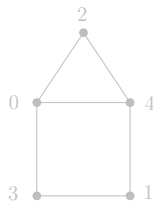
- $|f_{(2,1)}(v) - f_{(2,1)}(\tilde{v})| \geq 2$ for **neighboring** vertices v, \tilde{v} .
- $|f_{(2,1)}(v) - f_{(2,1)}(\tilde{v})| \geq 1$ for vertices v, \tilde{v} at **distance two**.

Graph coloring

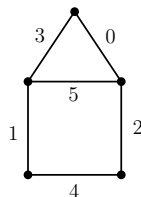
Vertex Coloring



Distance Vertex Labeling



Distance Edge Labeling



Definition (Edge-labeling mapping)

Edge-labeling mapping $f'_{(2,1)}: E \rightarrow \mathbb{N}_0$ satisfies:

- $|f'_{(2,1)}(e) - f'_{(2,1)}(\tilde{e})| \geq 2$ for **neighboring edges** e, \tilde{e} .
- $|f'_{(2,1)}(e) - f'_{(2,1)}(\tilde{e})| \geq 1$ for **edges** e, \tilde{e} at **distance two**.

Vertex problem definition

Definition (Minimal vertex-labeling of a graph)

$$\lambda_{(2,1)} := \min_{f_{(2,1)}} \max_{v \in V} f_{(2,1)}(v).$$

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Problem (Distance vertex-labeling of a graph G .)

Input: A graph G and a parameter λ .

Question: Is $\lambda_{(2,1)}(G) \leq \lambda$?

Previous results

Theorem (The dichotomy of distance vertex-labeling of a graph)
(Fiala, Kloks, Kratochvíl 2001)

The problem of distance vertex-labeling of a graph is:

- Polynomial-time solvable for $\lambda \leq 3$.

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- NP-complete for $\lambda \geq 4$.

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Theorem (The NP-completeness of chordal graphs.)
(Bodlaender, Kloks, Tan, van Leeuwen 2004)

It is NP-complete to decide whether $\lambda_{(2,1)}(G) \leq n$ for the chordal graph G on n vertices.

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- This version is **equivalent** to the vertex version on the class of **line-graphs**.

Our result

Theorem (The dichotomy of distance edge-labeling of a graph)
(Knop, TM 2015)

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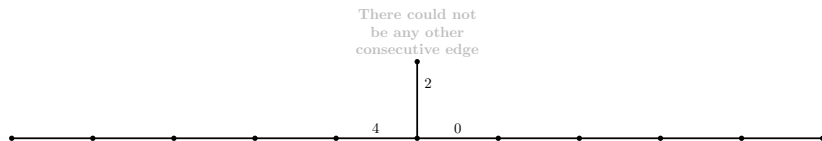
- Polynomial-time solvable for $\lambda \leq 4$.
- NP-complete for $\lambda \geq 5$.

Section 2

Polynomial cases

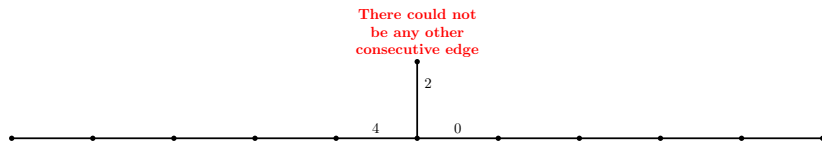
Polynomial cases ($\lambda \leq 4$)

An Example of the case analysis



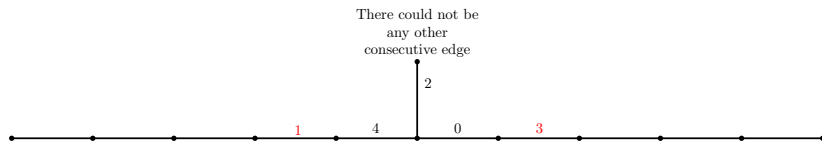
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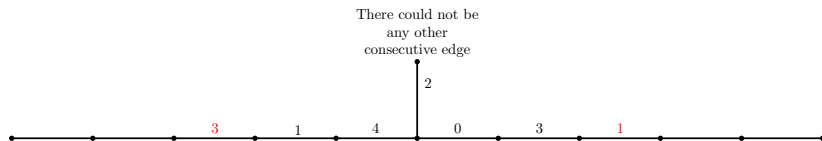
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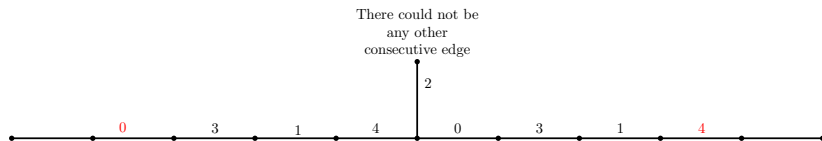
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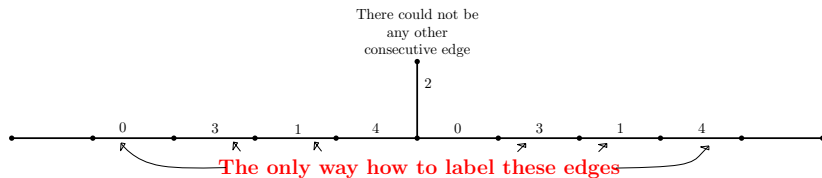
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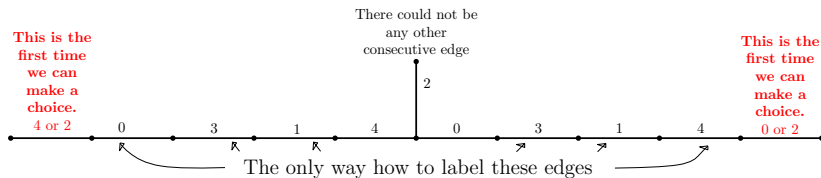
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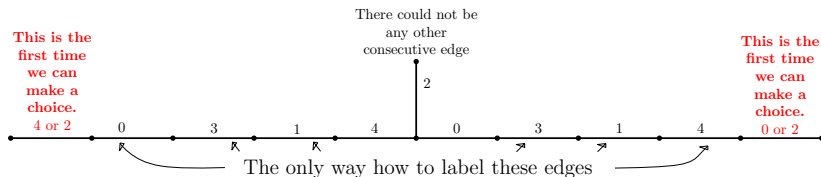
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- By a similar analysis the exact characterisation of graphs admitting **polynomial-time algorithm** is given in the paper.

Polynomial cases ($\lambda \leq 4$)

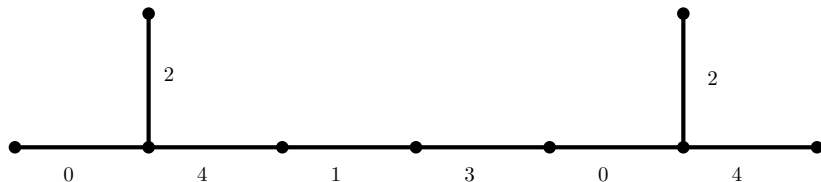
Characterisation

- All **paths** and **cycles**.

Polynomial cases ($\lambda \leq 4$)

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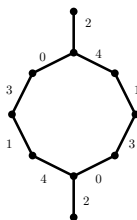
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Polynomial cases ($\lambda \leq 4$)

Characterisation

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- Some **generalised cycles** according to a parity between vertices of degree three.



Section 3

NP-completeness

Monotone not all equal 3-SAT

Problem (Monotone not all equal 3-SAT)

Input: A 3-MCNF formula φ .

Question: Is it possible to find an assignment such that each clause has at least one literal **set to true** and at least one literal **set to false**?

Monotone not all equal 3-SAT

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An Example clause

A clause in 3-MCNF (i.e. 3-CNF **without negations**):

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee x_4 \vee x_5).$$

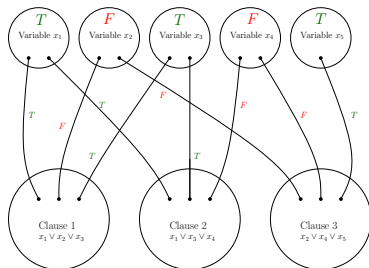
And an assignment: $x_1 = T$; $x_2 = F$; $x_3 = T$; $x_4 = F$; $x_5 = T$.

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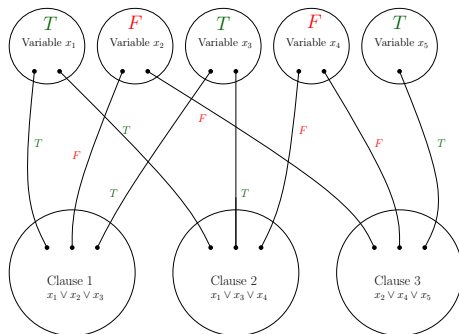
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The reduction

Interesting information about reductions

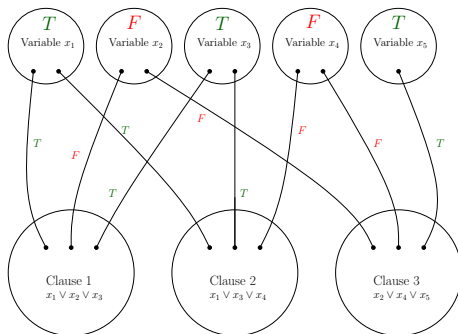
- The reduction has **four independent** parts.



The reduction

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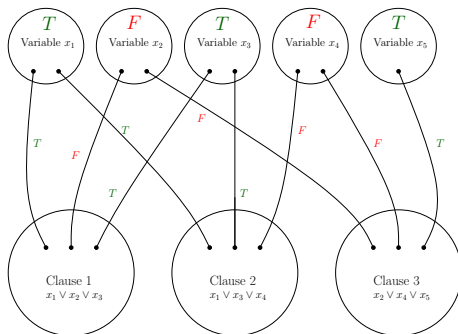
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 - It was necessary for us to distinguish cases for **even** and **odd** λ .



The reduction

Interesting information about reductions

- The reduction has **four independent** parts.
 - It was necessary for us to distinguish cases for **even** and **odd** λ .
 - And solve cases for small $\lambda = 5, 6$ by different reductions.



Section 4

Conclusion

Consequences

Corollaries of our theorem

- The distance edge-labeling problem **cannot be approximated** within a factor better than $\frac{6}{5}$.

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Corollaries of our theorem

- The distance edge-labeling problem **cannot be approximated** within a factor better than $\frac{6}{5}$.
- The distance edge-labeling problem is **paraNP-hard**.

Open problems

- **Simpler** reductions.

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Thank you for your attention!