Flexibility of Planar Graphs

Sharpening the Tools to Get Lists of Size Four

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Related Definitions

List coloring

A list assignment for a graph G is a function that to each vertex $v \in V(G)$ assigns a set $L(v)$ of colors. An L-coloring ϕ is proper if $\phi(v) \in L(v)$ for all $v \in V(G)$ and $\phi(u) \neq \phi(v)$ for an edge u, v.

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The choosability of graph G is the minimum integer k such that G has an L-coloring for every assignment L of lists of size at least k.

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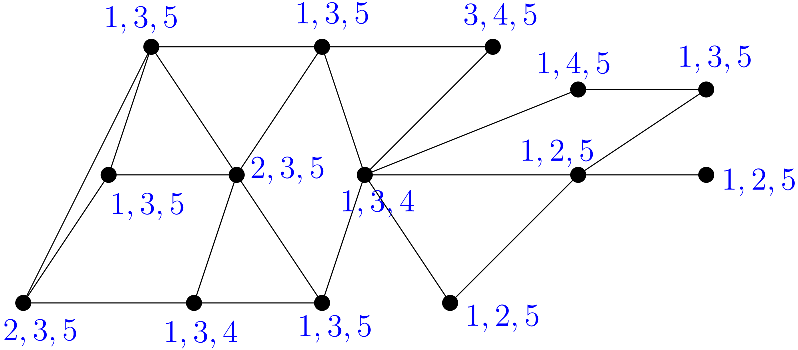
Precoloring Extension

The precoloring extension of graph G is a decision problem of extending a (pre)coloring of a subset of the vertices of a graph to a proper coloring of the whole graph.

Flexibility—sketch



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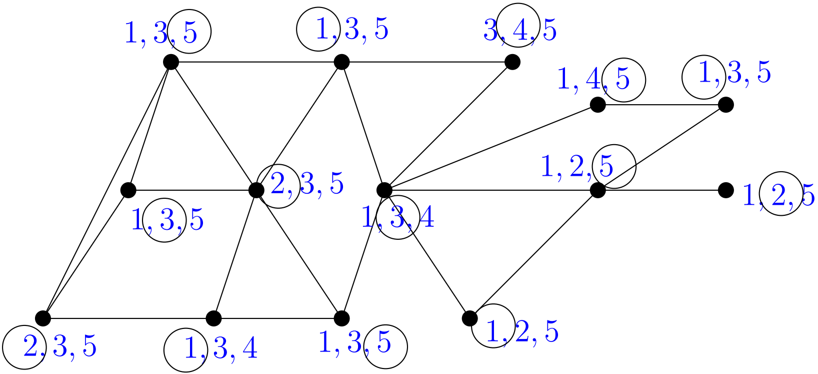

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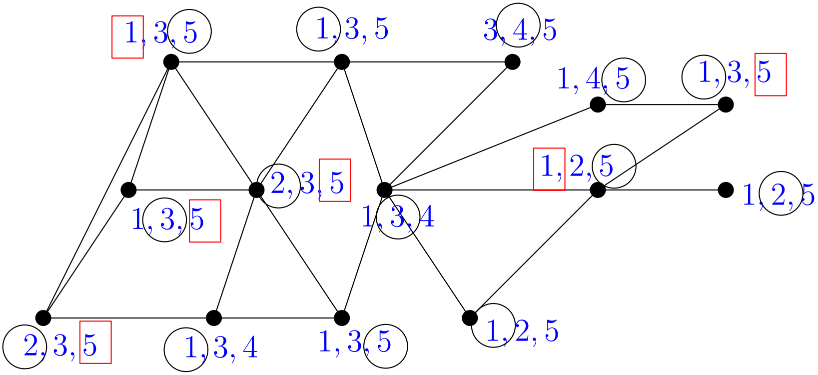

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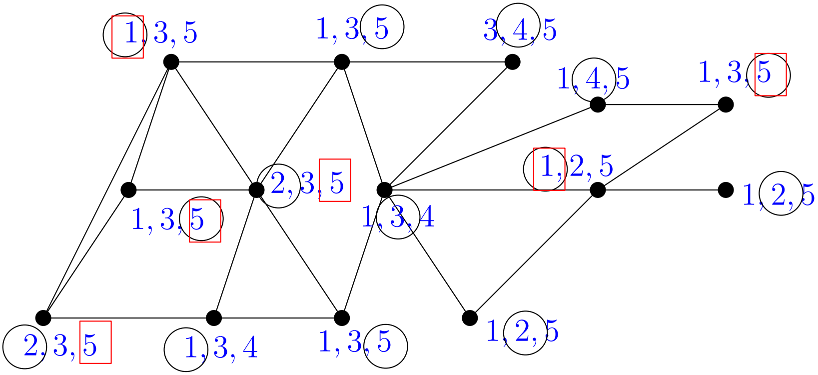

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A request for a graph G with a list assignment L is a function r with $\mathrm{dom}(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in \mathrm{dom}(r)$.

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ε-satisfiable request

For $\varepsilon > 0$, a request r is ε-satisfiable if there exists an L-coloring ϕ of G such that $\phi(v) = r(v)$ for at least $\varepsilon|\mathrm{dom}(r)|$ vertices $v \in \mathrm{dom}(r)$.

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ε-flexibility

We say that G with the list assignment L is ε-flexible if every request is ε-satisfiable.

Weighted flexibility

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Let L be a list assignment for a graph G. A weighted request is a function w that to each pair (v, c) with $c \in L(v)$ assigns a nonnegative real number.

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ε-satisfiable weighted request

Let $w(G, L) = \sum_{v \in V(G), c \in L(v)} w(v, c)$. For $\varepsilon > 0$, we say that w is ε-satisfiable if there exists an L-coloring ϕ of G such that

$$\sum_{v \in V(G)} w(v, \phi(v)) \geq \varepsilon w(G, L).$$

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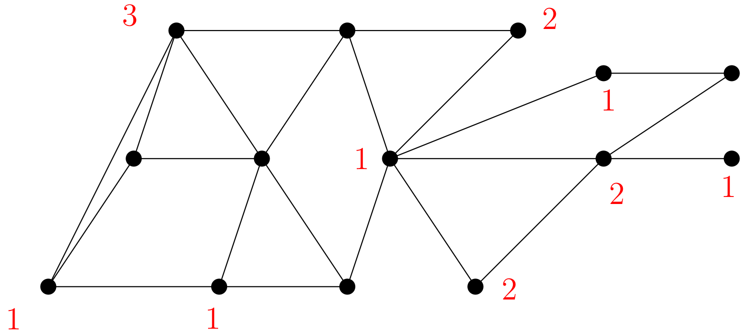

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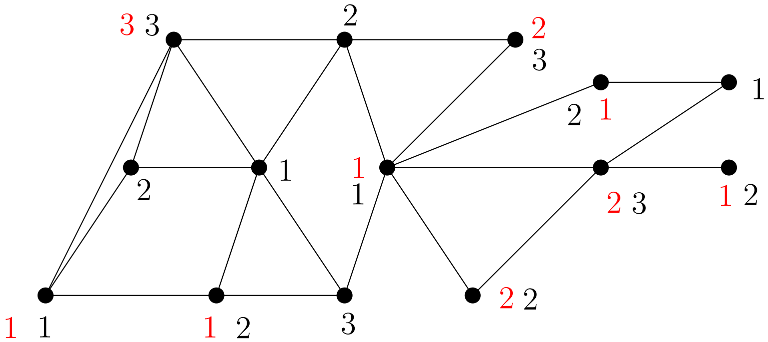

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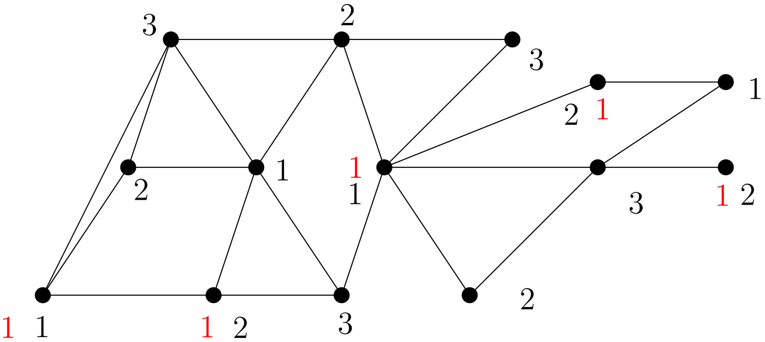

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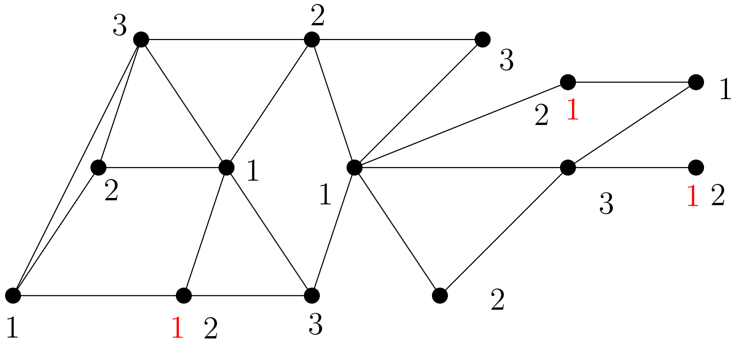

$\frac{1}{k}$ fraction of the remaining request is preserved.

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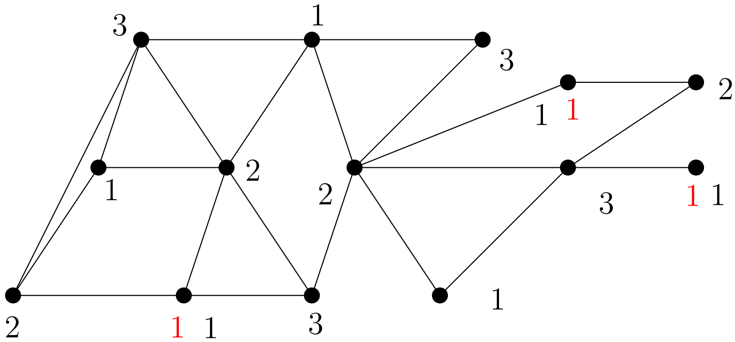

Questions

Meta-question

When choosability and flexibility bound coincide.

Conjecture (Dvořák, Norin, Postle JGT 2019)

There exists $\varepsilon > 0$ such that every planar graph with assignment of lists of size 5 is weighted ε-flexible.

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Theorem (T3 CCFHMM20+)

There exists $\varepsilon > 0$ such that every C_4-free, C_5-free, C_6-free planar graph is weakly ε-flexible for lists of size 4.

Related Results

Greedy algorithm gives choosability at most degeneracy $+1$. For flexibility [2] showed weighted ε-flexibility is at most degeneracy $+2$

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Deg.	5	3	4	3	3		3	4
Ch. lb.	5	4	4	3	3		4	4
Ch	5 [1]	4	4 [6]	3	3	4	4	5
W Fl.	6	4	5	4	3	4	4 T3	5
Fl.	6 [2]	4	5	4	3	4	5	5
w Fl	7 [2]	4 [3]	5 [5]	4	3 [4]	4 T2	5	5 T

1 Thomassen 94

2 Dvořák, Norin, Postle 19

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Linearity of Expectation

Lemma (DNP 19)

Let G be a graph and let L be a list assignment for G. Suppose G is L-colorable and there exists a probability distribution on L-colorings φ of G such that for every $v \in V(G)$ and $c \in L(v)$, $Prob[\varphi(v) = c] \geq \varepsilon$. Then G with L is weighted ε-flexible.

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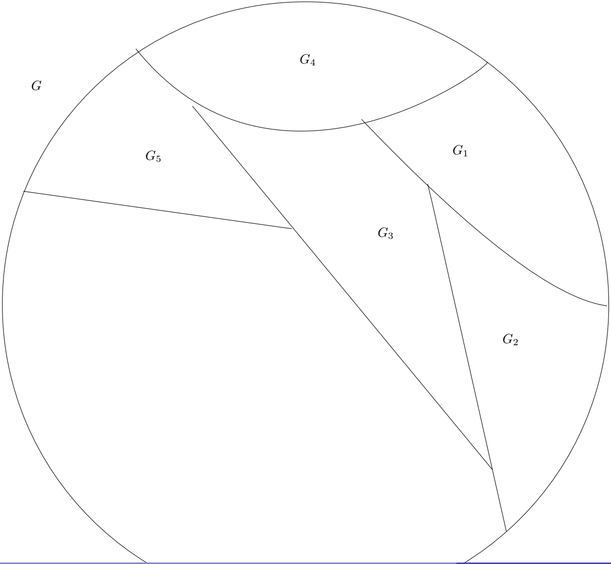

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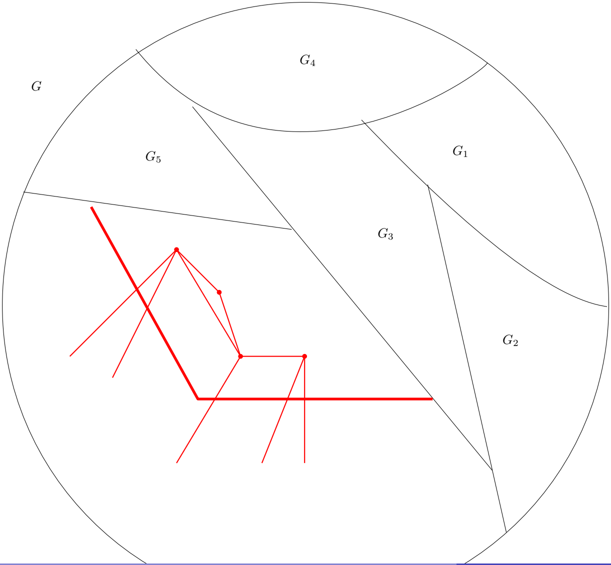

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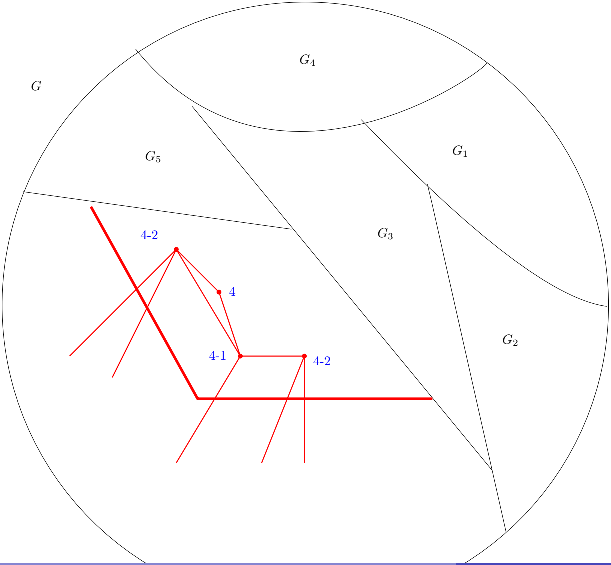

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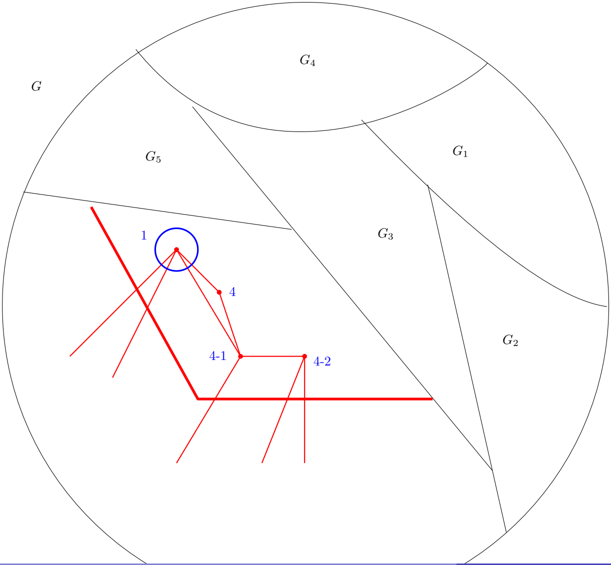

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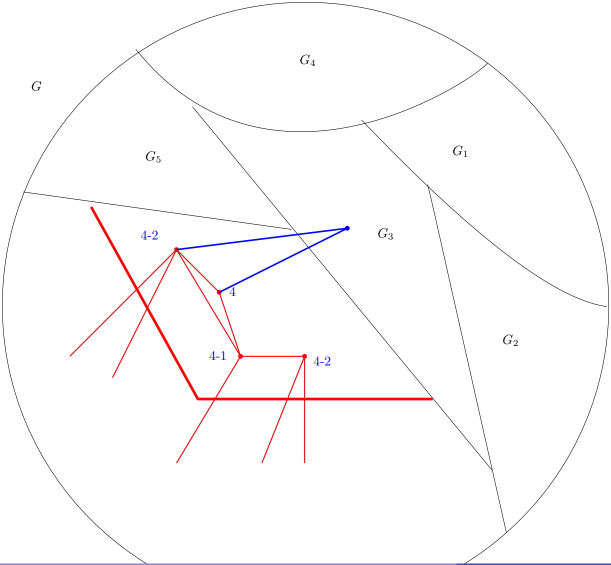

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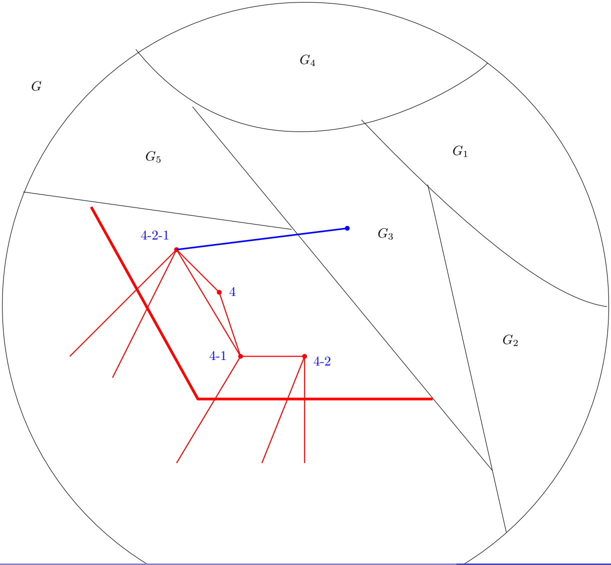

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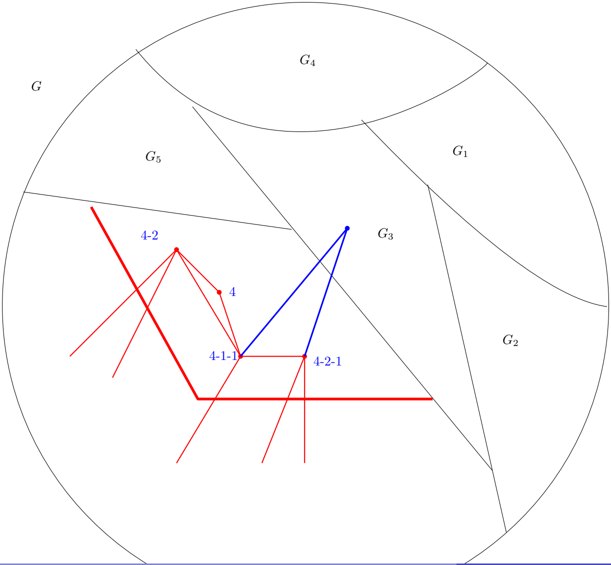

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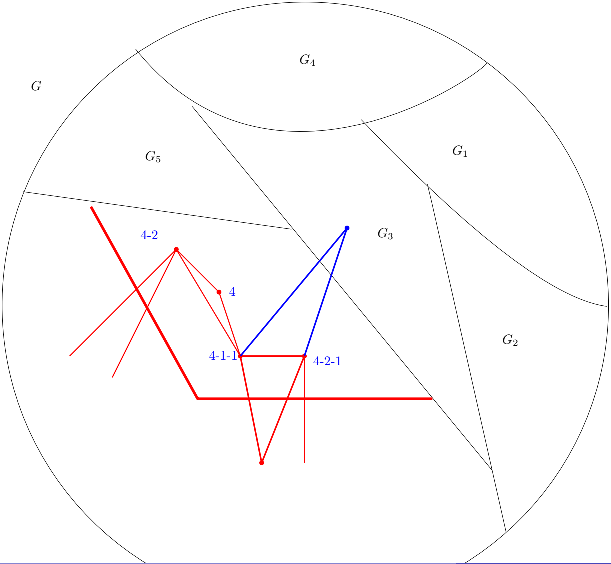

Reducible Configuration

Discharging. . .

Open cases

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Thank you for your attention!