Constricting the computational complexity gap of the 4-coloring problem in (P_t, C_3)-free graphs

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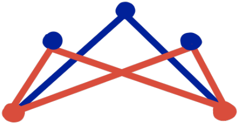
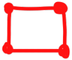


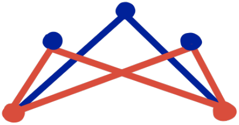
Cycles and Colourings
04 September, 2025

Notation

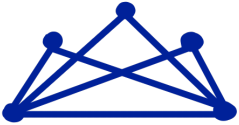
H-free graph: no copy of H as an induced subgraph.

H:





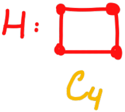
not H-free

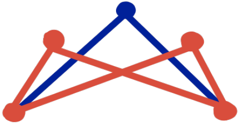


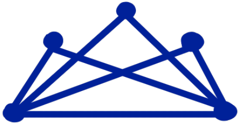
H-free

Notation

H*-free* graph: no copy of H as an induced subgraph.



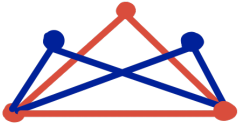




$(H_1, \ldots, H_k)$*-free* graph: H_1-free and... and H_k-free.

P_t: path on t vertices

C_t: cycle on t vertices



k-COLORING in H-free graphs

For every $k \geq 3$, k-COLORING on H-free graphs is NP-complete if H contains a cycle [Emden-Weinert, Hougardy, Kreuter 98] or

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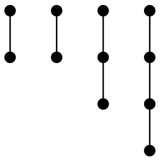
Hence, it remains to consider the case where every connected component of H is a path (i.e., H is a disjoint union of paths.)

Example (disjoint union of paths):



$$H = 2P_2 + P_3 + P_4$$

k-Coloring of P_t-free graphs

	k-Coloring of P_t-free graphs			
t	$k=3$	$k=4$	$k=5$	$k \geq 6$
$t \leq 5$	P	P	P	P
$t = 6$	P	P	NP-c	NP-c
$t = 7$	P	NP-c	NP-c	NP-c
$t \geq 8$	?	NP-c	NP-c	NP-c

'08 Hoang, Kaminski, Lozin, Sawada, Shu

'16 Huang

'04 Randerath, Schiermeyer

'19 Spirkl, Chudnovsky, Zhong

'17 Bonomo, Chudnovsky, Maceli, Schaudt, Stein, Zhong

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4-Coloring of (P_t,C_ℓ)-free graphs

	$\ell = 3$	$\ell = 4$[1]	$\ell \in \{5,6\}$[2]	$\ell = 7$[2]	$\ell \geq 8$[2]
$t \leq 6$[3]	P	P	P	P	P
$7 \leq t \leq 8$	?	P	NP-c	?	NP-c
$9 \leq t \leq 18$	?	P	NP-c	NP-c	NP-c
$19 \leq t \leq 21$	?	P	NP-c	NP-c	NP-c
$t \geq 22$	NP-c [4]	P	NP-c	NP-c	NP-c

[1] '14, Golovach, Paulusma, Song
[2] '17 Hell, Huang
[3] '21 ('24) Chudnovsky, Spirkl, Zhong
[4] '15 Huang, Johnson, Paulusma

4-Coloring of (P_t, C_ℓ)-free graphs

	$\ell = 3$	$\ell = 4$[1]	$\ell \in \{5,6\}$[2]	$\ell = 7$[2]	$\ell \geq 8$[2]
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REDUCTION BY

Huang, Johnson, Paulusma

4-COLORING on

(P_{22}, C_3)-free graphs

is NP-complete

MNAE 3-SAT

Monotone Not-All-Equal 3-SAT

$\times$ $\neg x_1$

only positive literals

$\times$ $1 \vee 1 \vee 1$
$0 \vee 0 \vee 0$

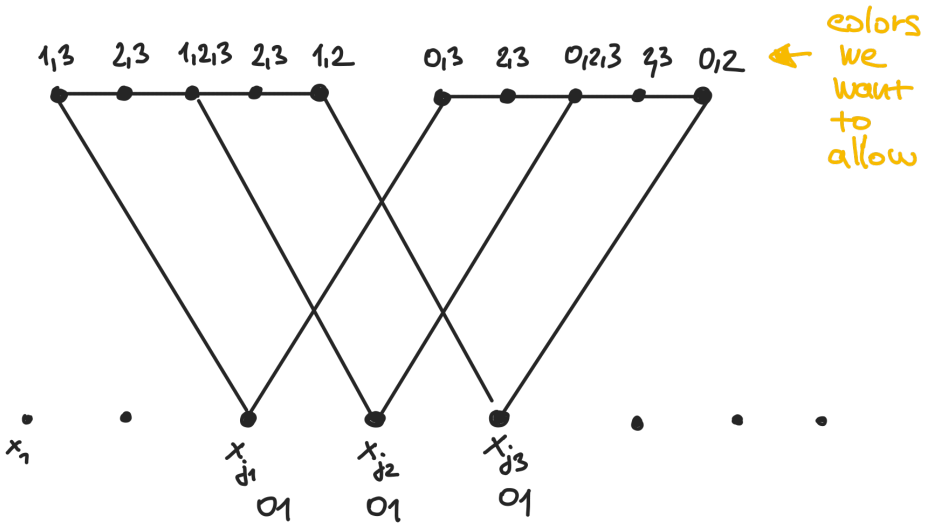
at least one true and one false

$\bigwedge_i (x_{i_1} \vee x_{i_2} \vee x_{i_3})$

each clause of three variables

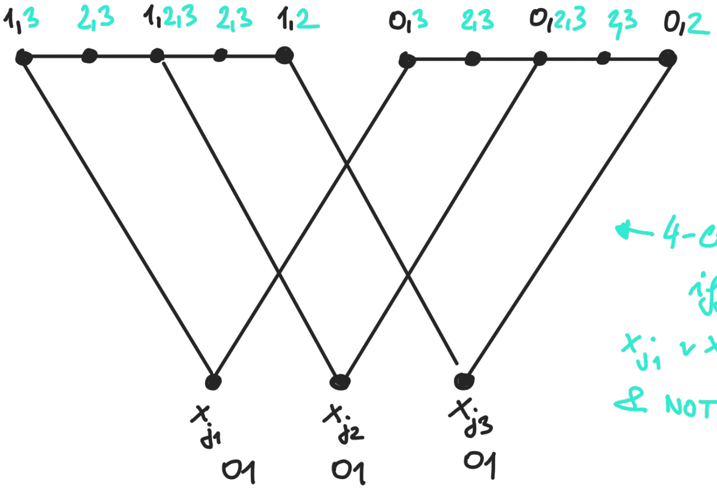
NP-complete ['78 Schaefer]

CLAUSE GADGET



clause $S_j = (x_{j1} \vee x_{j2} \vee x_{j3})$

CLAUSE GADGET

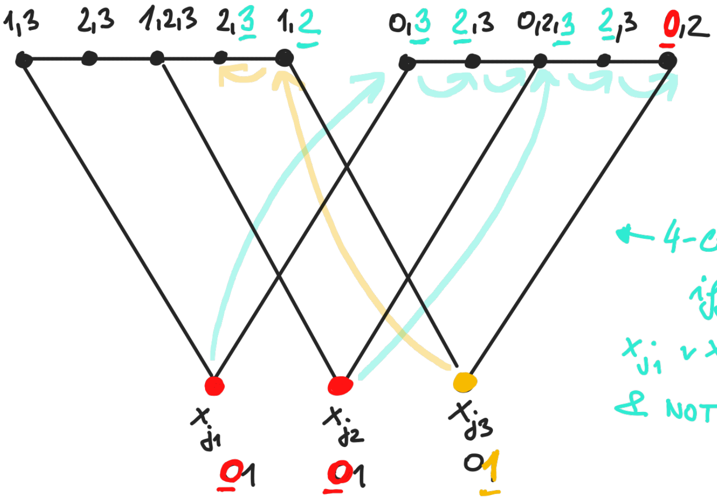


colors we want to allow

4-COLORABLE iff $x_{j1} \vee x_{j2} \vee x_{j3}$ TRUE & NOT 1,1,1 0,0,0

clause $S_j = (x_{j1} \vee x_{j2} \vee x_{j3})$

CLAUSE GADGET



colors we want to allow

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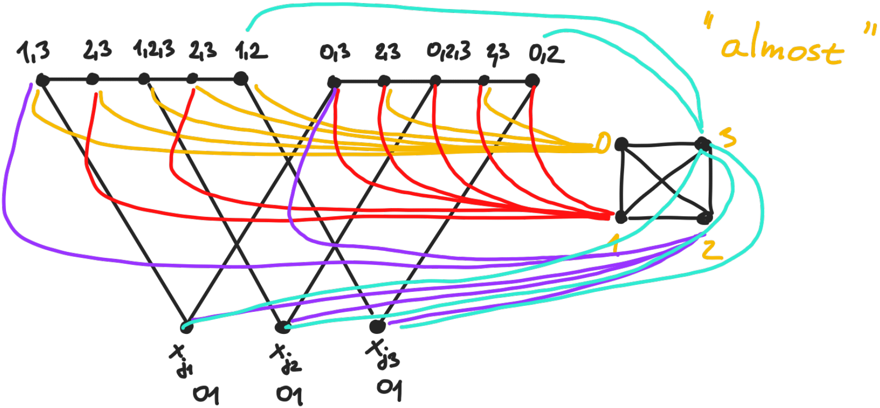
How to enforce the colors ?



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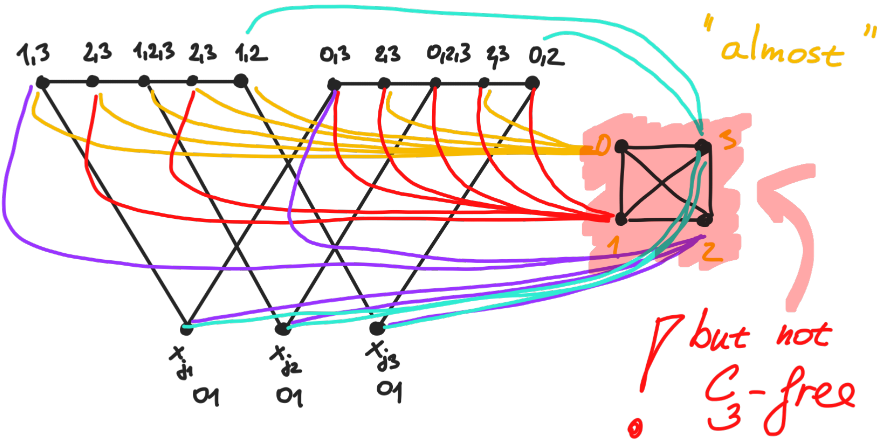

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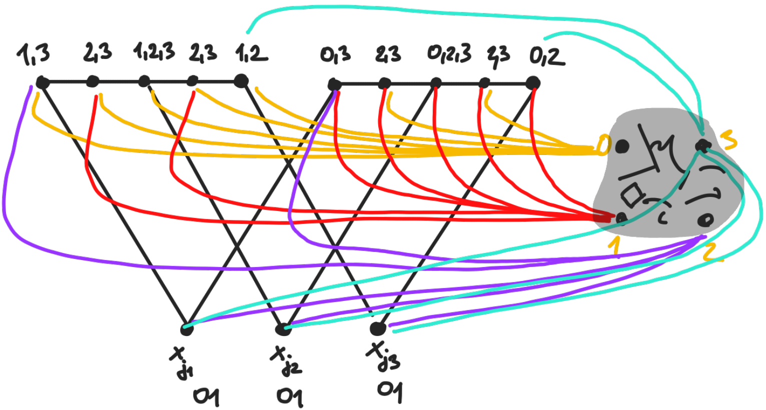
HOW TO ENFORCE THE COLORS ?



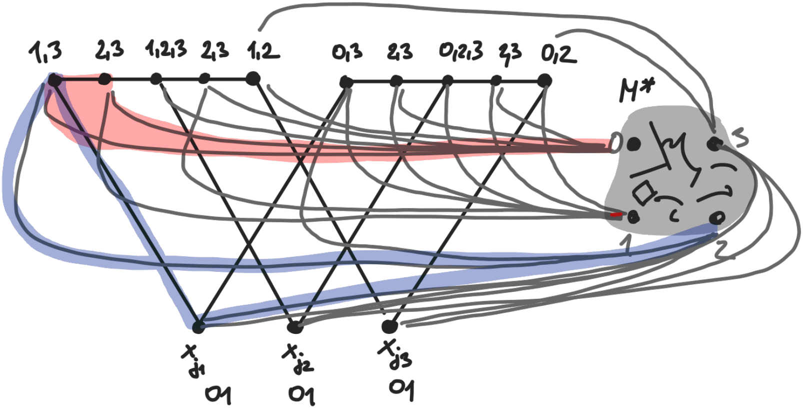
clause $S_j = (x_{j1} \vee x_{j2} \vee x_{j3})$

HOW TO ENFORCE THE 'ALLOWED' COLORS ?

5^{th} Mycielski graph without specific edge

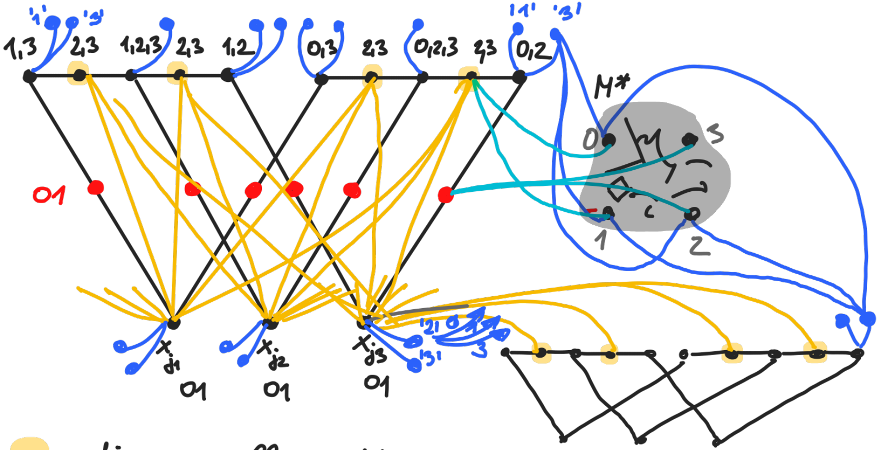
- 4-colorable
- C_3-free
- four 'special' vertices (in every coloring distinct colors)
- longest induced path ✓

? C_3 - FREE ?



... NOT YET

C_3 - FREE



: adjacent to all variables

: new vertices

: new 'pendant' vertices

P_{22} - FREE

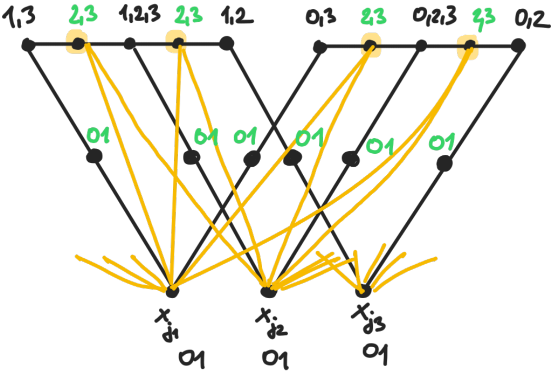
OUR RESULT

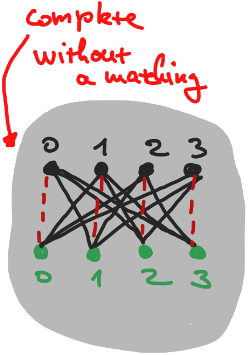
4-COLORING on

(P_{19}, C_3)-FREE

is NP-complete

CLAUSE GADGET



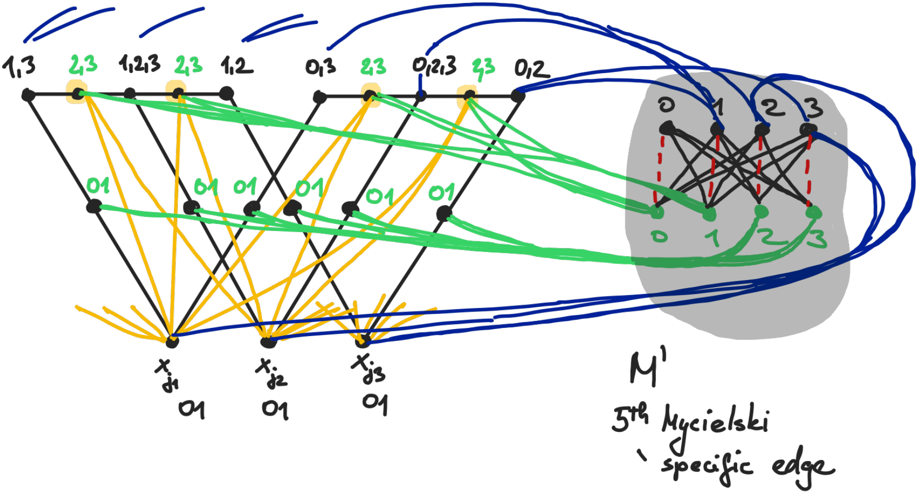


M'

5th Mycielski

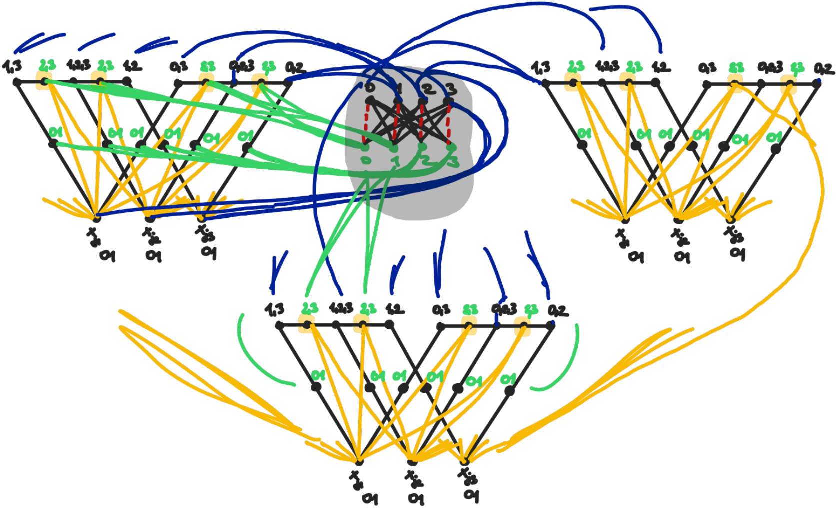
specific vertex

COLORS ENFORCEMENT

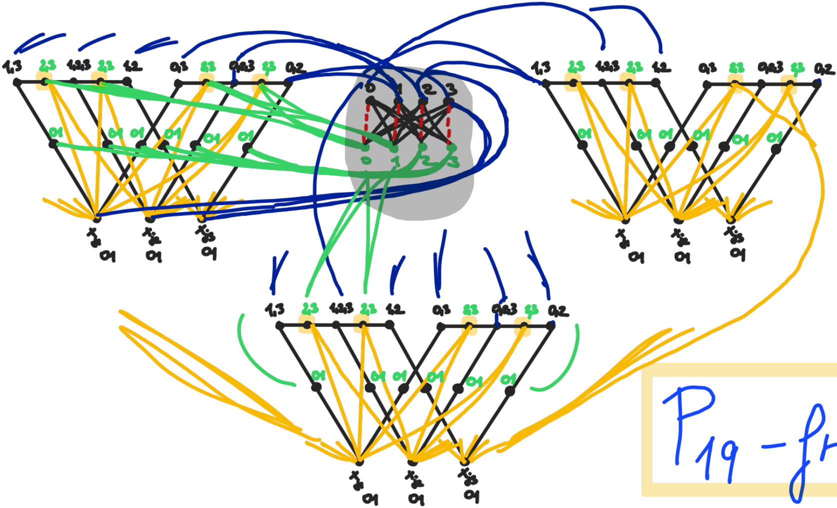


No pendant vertices

COLORS ENFORCEMENT



COLORS ENFORCEMENT

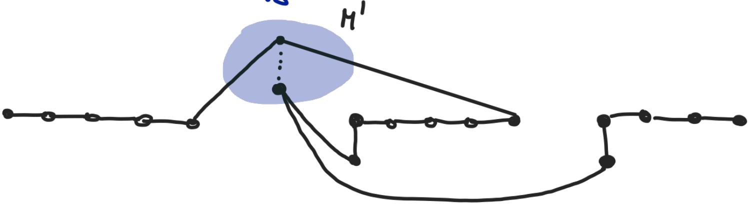


P_{19}-free

Computer assisted

FINAL COMMENTS

- Where is P_{18} ?



- Open: 4-coloring

  $(P_{7..18}, C_3)$-free

  $(P_{7..8}, C_7)$-free

Thank you for your attention!