Anti-Path Cover on Sparse Graph Classes

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MEMICS 2016,
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Section 1

Introduction

Problem definition

Anti-Hamiltonian path

Input:	A graph G.
Question:	Does the complement of a graph G has a path that contains all the vertices of G?

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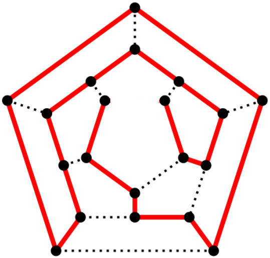

The tree-width

Definition (Tree decomposition)

A tree decomposition of a graph G is a pair (T, X), where $T = (I, F)$ is a tree, and $X = \{X_i \mid i \in I\}$ is a family of subsets of $V(G)$ (called bags) such that:

- the union of all X_i, $i \in I$ equals V,
- for all edges $\{v, w\} \in E$, there exists $i \in I$, such that $v, w \in X_i$ and
- for all $v \in V$ the set of nodes $\{i \in I \mid v \in X_i\}$ forms a subtree of T.

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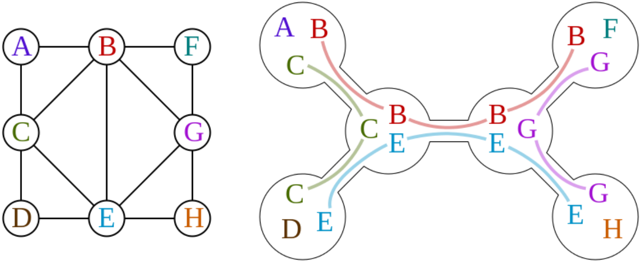

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- The closure $Clos(F)$ of a forest F is the graph obtained from F by making every vertex adjacent to all of its ancestors.
- The tree-depth, denoted as $\mathrm{td}(G)$, of a graph G is one more than the minimum height of a rooted forest F such that $G \subseteq Clos(F)$.

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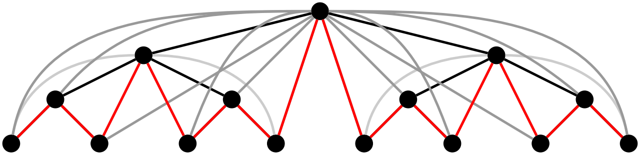

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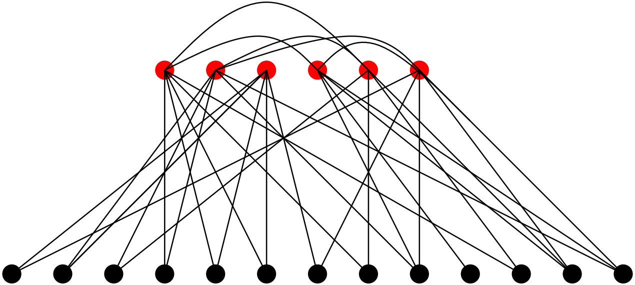

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The neighborhood diversity of a graph G is denoted by $\mathrm{nd}(G)$ and it is the minimum size of a partition of vertices into classes such that all vertices in the same class have the same neighborhood, i.e. $N(v) \setminus \{v'\} = N(v') \setminus \{v\}$, whenever v, v' are in the same class.

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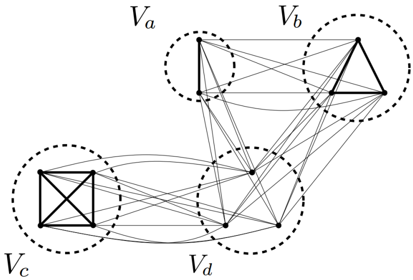

Our results

Theorem (An FPT algorithm on graph classes)
(Dvořák, Knop, TM 2016)

There is an FPT algorithm for the Anti-Hamiltonian path problem on graphs with

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- bounded tree-depth ($\mathrm{td}(G) \leq d$) neighborhood diversity bounded by $2^{2d} + d$.

Our results

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There is an FPT algorithm for the Anti-Hamiltonian path problem on graphs with linearly many edges (kn).

Section 2

FPT algorithm

Bondy-Chvátal closure

Theorem (Bondy-Chvátal closure)

Let $G = (V, E)$ be a graph of order $|V| \geq 3$ and suppose that u and v are distinct non-adjacent vertices such that $\deg(u) + \deg(v) \geq |V|$. Now G has a Hamiltonian path if and only if $(V, E \cup \{u, v\})$ has a Hamiltonian path.

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Theorem (BC closure complement version)

Let $G = (V, E)$ be a graph of order $|V| \geq 3$ and suppose that u and v are distinct adjacent vertices such that $\deg(u) + \deg(v) < |V|$. Now G has a Hamiltonian path if and only if $(V, E \setminus \{u, v\})$ has a Hamiltonian path.

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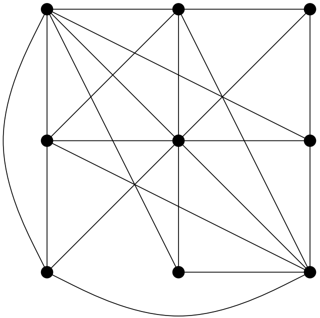

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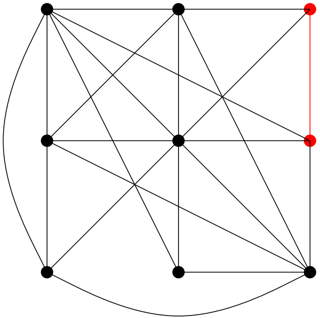

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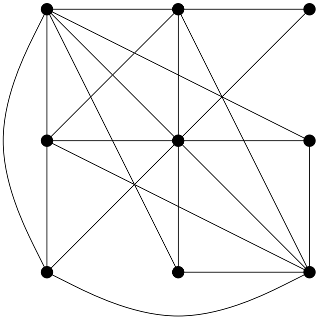

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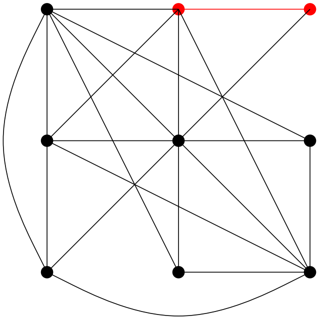

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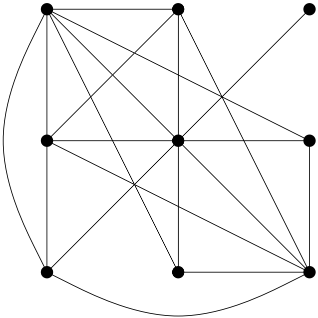
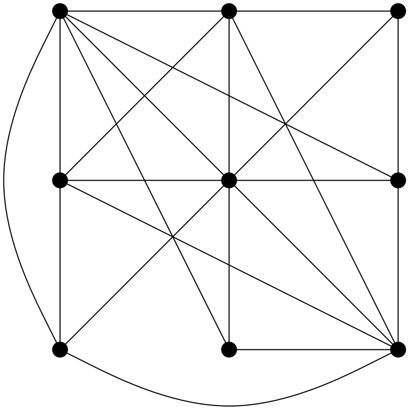


Sketch of the proof

- At most linearly many application of BC, each takes constant time.



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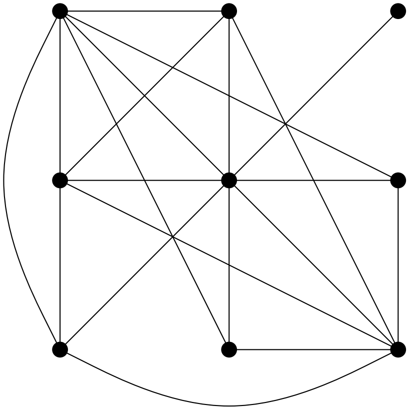

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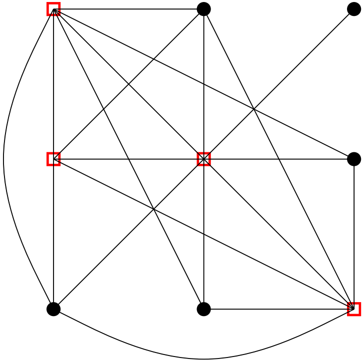

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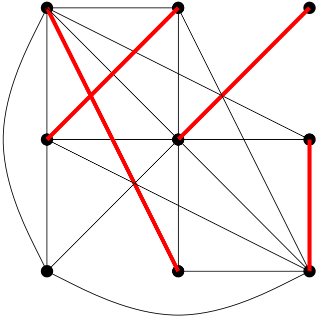

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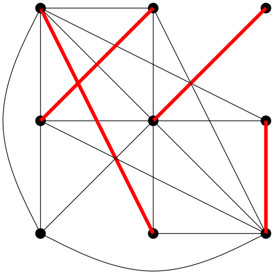

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- That is a contradiction since we have at most $\frac{kn}{2}$ edges.

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Section 3

Conclusions and further investigation

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- We showed that this problem admits an FPT algorithm even on wider class of graphs, not even sparse, even thou it is still NP-hard in general.

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Thank you for your attention!