# Parameterized complexity of fair deletion problems

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## Deletion problems

- In *edge deletion problem*, we are given a graph $G$ and a graph property $\pi$, the goal is to find a subset $F$ of edges such that $G \setminus F$ satisfies $\pi$.
- Similarly, in *vertex deletion problems*, the task is to find a subset $W$ of vertices such that $G \setminus W$ satisfies $\pi$.

### Examples

- Vertex Cover: Find set of vertices $W$ such that $G \setminus W$ is an *independent set*.
- Feedback Vertex set: Find set of vertices $W$ such that $G \setminus W$ is *acyclic*.
- Feedback Arc set: Find set of edges $F$ such that $G \setminus F$ is *acyclic*.
- Perfect Matching: Find set of edges $F$ such that $G \setminus F$ is 1-*regular*.

Usually (Vertex Cover, Feedback Arc Set, ...) just finding a set satisfying given property is trivial. In those cases, we aim to find *smallest* such set (or a set smaller than a given size).

### Fair deletion problems

In *fair deletion problems*, instead of finding small set, we require that the set is *locally small* in the following sense:

- For the *edge variant*, we aim to find set of edges $F$ such that $(V, F)$ has small degree (i.e. there is small number of deleted edges incident to each vertex).
- For the *vertex variant*, we want to find set of vertices $W$ such that $|N(v) \cap W|$ is small. (i.e. there is small number of deleted neighbors for each vertex).

## Graph properties and Logic

Our aim is to study all reasonable properties at once; in the spirit of Courcelle's theorem, we consider all properties expressible in *Monadic second order logic*.
Monadic second order logic is an extension of First Order logic where we can additionally *quantify over sets of elements*.
There are two versions of MSO logic for graphs, depending on which structure we use to represent a graph:

### Monadic Second Order logic

$\mathsf{MSO}_1$: The graph is described by its vertex set and a binary adjacency relation $adj(\cdot,\cdot)$.
$\mathsf{MSO}_2$: The graph is described by its vertex set, its edge set and a binary *incidence relation* $inc(\cdot,\cdot)$.
Since *adj* can be defined in terms of *inc*, $\mathsf{MSO}_2$ is a generalization of $\mathsf{MSO}_1$, as we can additionally quantify over edges and sets of edges.
It is known that $\mathsf{MSO}_2$ is strictly stronger than $\mathsf{MSO}_1$: the property that a graph is Hamiltonian can be expressed in $\mathsf{MSO}_2$ but not in $\mathsf{MSO}_1$

## Generalization of deletion problems

Kolman et al. considered a generalization of deletion problems: instead of requiring that $G \setminus F \models \psi$, we require that $G \models \psi'(F)$.
This way we can still impose conditions on $G \setminus F$, but additionally, we can impose conditions on $F$ itself.
This can be used to describe for example Matching-Cut.

## Previous results

- NP-completeness of Fair feedback edge set by Lin and Sahni.
- $\mathcal{O}(\sqrt{n})$-approximation for Fair Odd Cycle Transversal by Kolman et al.
- $n^{\mathcal{O}(\mathrm{tw}(G))}$ algorithm for the generalized version by Kolman et al. (the only previous result in the metatheorem context).

## Considered graph parameters

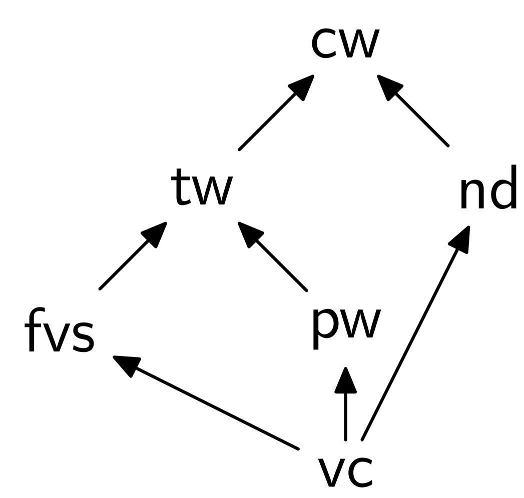
- Every problem considered is parameterized by $|\psi|$ (this is necessary for every metatheorem)

### Structural parameters

- $\mathrm{tw}(\mathbf{G})$ – the *tree-width* of a graph $G$
- $\mathrm{pw}(\mathbf{G})$ – the *path-width* of a graph $G$
- $\mathrm{fvs}(\mathbf{G})$ – the size of *minimum feedback vertex set* of a graph $G$
- $\mathrm{vc}(\mathbf{G})$ – the size of *minimum vertex cover* of a graph $G$
- $\mathrm{nd}(\mathbf{G})$ – the *neighborhood diversity* of a graph $G$

### Parameter relationship

Arrow from parameter $p$ to $q$ means that $p$ can be bounded by a function of $q$ (in other words, $p$ is more restrictive).




## Our results

### Hardness

- The generalized version of fair deletion problem (both vertex and edge variant) is $\mathsf{W}[1]$-hard for combined parameterization by $|\psi|$, $\mathrm{fvs}(G)$, and $\mathrm{pw}(G)$.
- There is no algorithm with running time $f(|\psi|, k)n^{o(\sqrt{k})}$ (where $k = \mathrm{fvs}(G) + \mathrm{pw}(G)$), unless Exponential Time Hypothesis fails.

### FPT algorithms

- The generalized $\mathsf{MSO}_1$ version is $\mathsf{FPT}$ with respect to $\mathrm{nd}(G)$. ($\mathsf{MSO}_2$ version is hopeless, since $\mathsf{MSO}_2$ model checking is hard even on cliques).
- The generalized $\mathsf{MSO}_2$ version is $\mathsf{FPT}$ with respect to $\mathrm{vc}(G)$.

## Proof sketch

We reduce Equitable MSO partition to both Fair Vertex Deletion and Fair Edge deletion. Hardness follows from hardness of particular instances of Equitable MSO partitions: one can use Equitable Coloring or Equitable Connected Partition.

## Open questions

- Can be the lower bounds extended to the non-generalized variant of deletion problems?
- Can the lower bound $f(|\psi|, k)n^{o(\sqrt{k})}$ be improved?

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