Parameterized complexity of fair deletion problems
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Deletion problems
- In edge deletion problem, we are given a graph $G$ and a graph property $\pi$, the goal is to find a subset $F$ of edges such that $G\setminus F$ satisfies $\pi$.
- Similarly, in vertex deletion problems, the task is to find a subset $W$ of vertices such that $G\setminus W$ satisfies $\pi$.

Examples
- **VERTEX COVER**: Find set of vertices $W$ such that $G \setminus W$ is an independent set.
- **Feedback Vertex Set**: Find set of vertices $W$ such that $G \setminus W$ is acyclic.
- **Feedback Arc Set**: Find set of edges $F$ such that $G \setminus F$ is acyclic.
- **Perfect Matching**: Find set of edges $F$ such that $G \setminus F$ is 1-regular.

Graph properties and Logic
Our aim is to study all reasonable properties at once; in the spirit of Courcelle’s theorem, we consider all properties expressible in Monadic Second Order logic. Monadic second order logic is an extension of First Order logic where we can additionally quantify over sets of elements. There are two versions of MSO logic for graphs, depending on which structure we use to represent a graph:

**MSO**: The graph is described by its vertex set and a binary incidence relation $\text{inc}(\cdot, \cdot)$.
**MSO$_2$**: The graph is described by its vertex set, its edge set and a binary adjacency relation $\text{adj}(\cdot, \cdot)$. Since $\text{adj}$ can be defined in terms of $\text{inc}$, MSO$_2$ is a generalization of MSO, as we can additionally quantify over edges and sets of edges. It is known that MSO$_2$ is strictly stronger than MSO$_1$; the property that a graph is Hamiltonian can be expressed in MSO$_2$ but not in MSO$_1$.

Generalization of deletion problems
Kolman et al. considered a generalization of deletion problems: instead of requiring that $G \setminus F \models \psi$, we require that $G \models \psi(F)$. This way we can still impose conditions on $G \setminus F$, but additionally, we can impose conditions on $F$ itself.

This can be used to describe for example Matching-Cut.

Previous results
- NP-completeness of Fair Feedback Edge set by Lin and Sahni.
- $O(\sqrt{m})$-approximation for Fair Odd Cycle Transversal by Kolman et al.
- $\psi(\text{inc}(G))$ algorithm for the generalized version by Kolman et al. (the only previous result in the metatheorem context).

Considered graph parameters
- Every problem considered is parameterized by $|\psi|$ (this is necessary for every metatheorem).

Structural parameters
- $\text{tw}(G)$ – the tree-width of a graph $G$.
- $\text{pw}(G)$ – the path-width of a graph $G$.
- $\text{fvs}(G)$ – the size of minimum feedback vertex set of a graph $G$.
- $\text{vc}(G)$ – the size of minimum vertex cover of a graph $G$.
- $\text{nd}(G)$ – the neighborhood diversity of a graph $G$.

Parameter relationship
Arrow from parameter $p$ to $q$ means that $p$ can be bounded by a function of $q$ (in other words, $p$ is more restrictive).

Our results

Hardness
- The generalized version of fair deletion problem (both vertex and edge variant) is $W[1]$-hard for combined parameterization by $|\psi|$, $\text{fvs}(G)$, and $\text{pw}(G)$.
- There is no algorithm with running time $f(|\psi|, k)\cdot n^{o(k)}$ (where $k = \text{fvs}(G) + \text{pw}(G)$), unless Exponential Time Hypothesis fails.

FPT algorithms
- The generalized MSO$_2$ version is FPT with respect to $\text{nd}(G)$. (MSO$_2$ version is hopelessly, since MSO$_2$ model checking is hard even on cliques.)
- The generalized MSO$_2$ version is FPT with respect to $\text{vc}(G)$.

Proof sketch
We reduce Equitable MSO partition to both Fair Vertex Deletion and Fair Edge Deletion. Hardness follows from hardness of particular instances of Equitable MSO partitions: one can use Equitable Coloring or Equitable Connected Partition.

Open questions
- Can the lower bounds extended to the non-generalized variant of deletion problems?
- Can the lower bound $f(|\psi|, k)n^{o(k)}$ be improved?

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