Clique-Width: Harnessing the Power of Atoms

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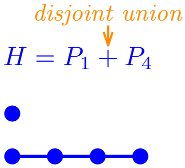
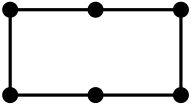
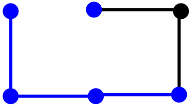
- *H-free* if it does not contain H as an induced subgraph
- *(H_1, H_2)-free* if it is H_1-free and H_2-free.





$(P_1 + P_4)$-free



not $(P_1 + P_4)$-free

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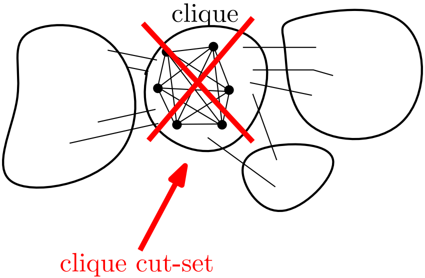
An *atom* is a connected graph with no clique cut-set



Definitions: Clique-width

The *clique-width* of a graph is the minimum number of labels required to construct the graph using the operations:

- create a new graph from a single vertex with a label,
- take the disjoint union of two labelled graphs,
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A graph class has *bounded clique-width* if there exists a constant c which upper-bounds the clique-width of each graph in the class.

Clique-width and atoms

- **bounded clique-width**
  $\rightsquigarrow$ polynomial algorithm for large collection of NP-hard problems
  (problems definable in MSO_1—*(Courcelle, Makowsky, Rotics)*, . . .)

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- **atoms + hereditary class $\mathcal{C}$**
  some problems polynomial time solvable on the atoms of $\mathcal{C}$
  $\rightsquigarrow$ polynomial-time solvable on graph class $\mathcal{C}$

  e.g., Coloring, Maximum Independent Set, Maximum Clique, Maximum Induced Matching, . . .

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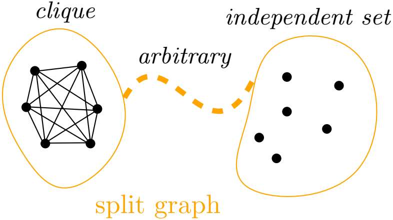

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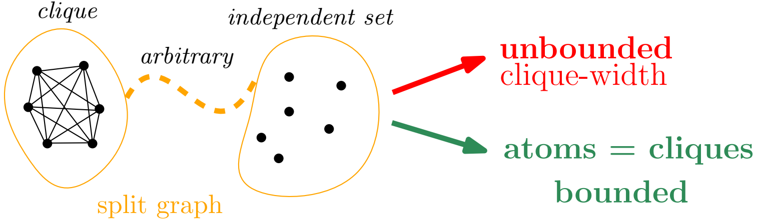

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  - YES: Every (C_4, P_6)-free atom has clique-width at most 18 (*Gaspers, Huang, Paulusma, 2019*).

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Our goal: Which (H_1,H_2)-free graph classes of unbounded clique-width have the property that their atoms have bounded clique-width?

Our results

The class of (H_1,H_2)-free atoms has bounded clique-width if
$H_1 \subseteq_i 2P_2$ and $H_2 \subseteq_i \overline{P_2 + P_3}$.

induced subgraph

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The class of (H_1,H_2)-free atoms has unbounded clique-width if
$H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$
$H_1 \notin \overline{\mathcal{S}}$ and $H_2 \notin \overline{\mathcal{S}}$
$H_1 \supseteq_i K_3 + P_1$ and $H_2 \supseteq_i 4P_1$ or $2P_2$
$H_1 \supseteq_i K_{1,3}$ and $H_2 \supseteq_i K_4$ or C_4
$H_1 \supseteq_i \overline{2P_1 + P_2}$ and $H_2 \supseteq_i K_{1,3}, 5P_1,\ P_2 + P_4$ or $P_1 + P_6$
$H_1 \supseteq_i 2P_1 + P_2$ and $H_2 \supseteq_i K_3 + P_1, K_5,\ \overline{P_2 + P_4}$ or $\overline{P_1 + P_6}$
$H_1 \supseteq_i K_3$ and $H_2 \supseteq_i 2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2$ or $2P_3$
$H_1 \supseteq_i 3P_1$ and $H_2 \supseteq_i \overline{2P_1 + 2P_2}, \overline{2P_1 + P_4}, \overline{4P_1 + P_2}, \overline{3P_2}$ or $\overline{2P_3}$
$H_1 \supseteq_i K_4$ and $H_2 \supseteq_i P_1 + P_4, 3P_1 + P_2$ or $2P_2$
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$H_1 \supseteq_i P_1 + 2P_2$ or P_6 and $H_2 \supseteq_i \overline{P_1 + 2P_2}$ or $\overline{P_2 + P_3}$.

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$H_1 \supseteq_i K_{1,3}$ and $H_2 \supseteq_i K_4$ or C_4
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$H_1 \supseteq_i P_1 + 2P_2$ or P_6 and $H_2 \supseteq_i \overline{P_1 + 2P_2}$ or $\overline{P_2 + P_3}$.

induced subgraph

Clique-width classified for all but $10 + 12 = 22$ (H_1,H_2)-free atoms.

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☹ removing vertices

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(C_4, P_5)-free **atoms: bounded** clique-width (*known*)

$(\overline{C_4}, \overline{P_5})$-free **atoms: unbounded** clique-width

Results: bounded clique-width of atoms

$(2P_2, \overline{P_2 + P_3})$

"*paraglider*"

Theorem

The class of (, )-free atoms have bounded clique-width (whereas the class of (, )-free graphs has unbounded clique-width).

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Proof sketch:

(i) $(2P_2, \overline{P_2 + P_3})$-free atoms with an induced C_5,

(ii) $(C_5, 2P_2, \overline{P_2 + P_3})$-free atoms with an induced C_4,

(iii) $(C_4, C_5, 2P_2, \overline{P_2 + P_3})$-free atoms (known to have bounded clique-width).

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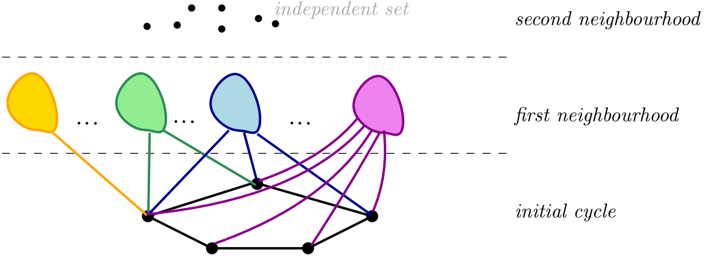

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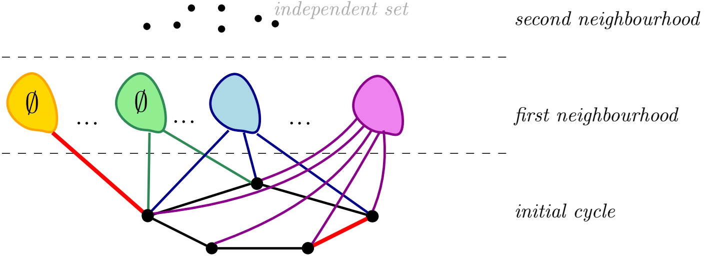

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Proof sketch: *Where atoms helped?*

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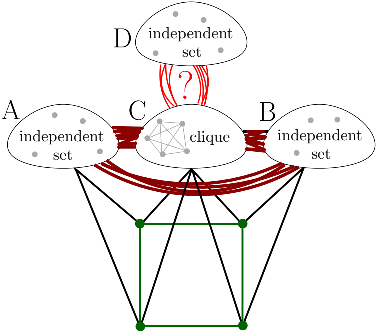

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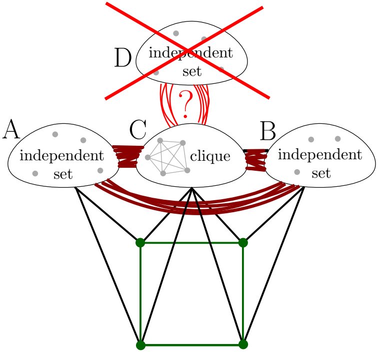

Open Problems

Does the class of (H_1,H_2)*-free atoms have bounded clique-width if*

(i) $H_1 = P_6$ *and* $H_2 \in \{\overline{2P_1 + P_2}, \overline{P_1 + P_4}\}$

(v) $H_1 = \overline{P_6}$ *and* $H_2 \in \{2P_1 + P_2, P_1 + P_4\}$

(ii) $H_1 = C_4$ *and* $H_2 \in \{P_1 + 2P_2, P_2 + P_4, 3P_2\}$

(iii) $H_1 = \overline{P_1 + 2P_2}$ *and* $H_2 \in \{2P_2, P_2 + P_3, P_5\}$

(iv) $H_1 = \overline{P_2 + P_3}$ *and* $H_2 \in \{P_2 + P_3, P_5\}$

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*(vi) $H_1 = K_3$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$

*(vii) $H_1 = 3P_1$ *and* $H_2 \in \{\overline{P_1 + S_{1,1,3}}, \overline{S_{1,2,3}}\}$

*(viii) $H_1 = \overline{2P_1 + P_2}$ *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$

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*(x) $H_1 = \overline{P_1 + P_4}$ *and* $H_2 = P_2 + P_3$ *or*

*(xi) $H_1 = P_1 + P_4$ *and* $H_2 = \overline{P_2 + P_3}$.

Thank you for your attention!