Path Eccentricity and Forbidden Induced Subgraphs

Sylwia Cichacz Claire Hilaire Tomáš Masařík Jana Masaříková Martin Milanič

University of Warsaw, Poland

EUROCOMB Budapest 2025

Path Eccentricity

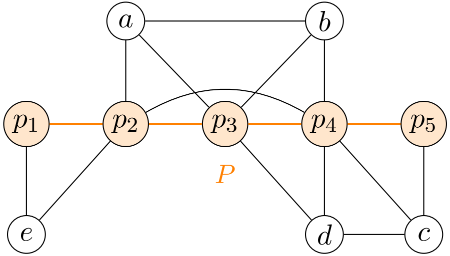
Definition (Path Eccentricity)

For a connected graph G, the *path eccentricity* pe**(G)** is the **minimum** $k \geq 0$ s.t. there exists a **path** P with

$$\max_{v \in V(G)} \mathrm{dist}(v, P) \leq k.$$

- $\mathrm{pe}(G) = 0 \Leftrightarrow G$ has a **Hamiltonian path**.
- $\mathrm{pe}(G) \leq 1 \Leftrightarrow G$ has a **dominating path**.

Note: P does not need to be induced.



All vertices are within distance **1** from P
$\Rightarrow \mathrm{pe}(G) \leq$ **1**.

Path Eccentricity

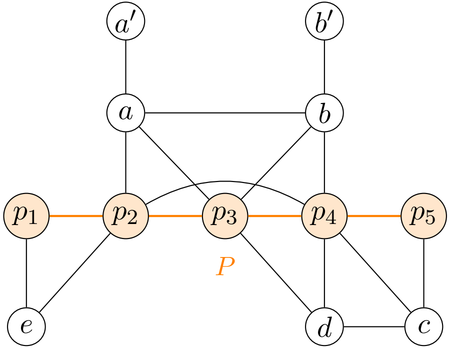
Definition (Path Eccentricity)

For a connected graph G, the *path eccentricity* $\mathrm{pe}(\mathbf{G})$ is the **minimum** $k \geq 0$ s.t. there exists a **path** P with

$$\max_{v \in V(G)} \mathrm{dist}(v, P) \leq k.$$

- $\mathrm{pe}(G) = 0 \Leftrightarrow G$ has a **Hamiltonian path**.
- $\mathrm{pe}(G) \leq 1 \Leftrightarrow G$ has a **dominating path**.

Note: P does not need to be induced.



All vertices are within distance **2** from P
$\Rightarrow \mathrm{pe}(G) \leq \mathbf{2}$.

Path Eccentricity

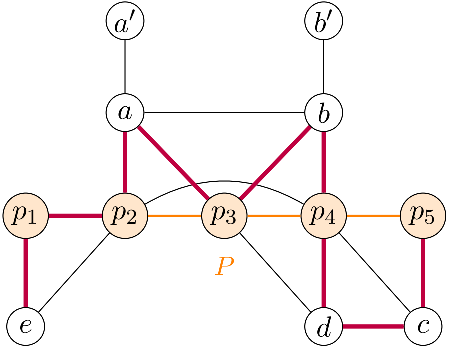
Definition (Path Eccentricity)

For a connected graph G, the *path eccentricity* pe**(G)** is the **minimum** $k \geq 0$ s.t. there exists a **path** P with

$$\max_{v \in V(G)} \mathrm{dist}(v, P) \leq k.$$

- $\mathrm{pe}(G) = 0 \Leftrightarrow G$ has a **Hamiltonian path**.
- $\mathrm{pe}(G) \leq 1 \Leftrightarrow G$ has a **dominating path**.

Note: P does not need to be induced.



All vertices are within distance **1** from P

$\Rightarrow \mathrm{pe}(G) \leq$ **1**.

Path Eccentricity

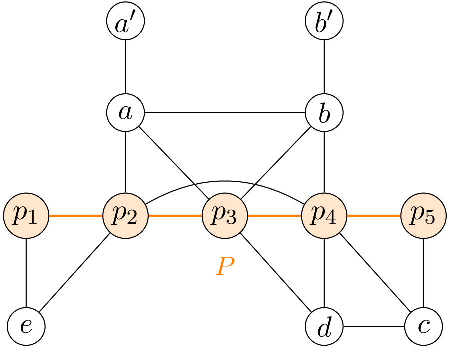
Definition (Path Eccentricity)

For a connected graph G, the *path eccentricity* pe**(G)** is the **minimum** $k \geq 0$ s.t. there exists a **path** P with

$$\max_{v \in V(G)} \mathrm{dist}(v, P) \leq k.$$

- $\mathrm{pe}(G) = 0 \Leftrightarrow G$ has a **Hamiltonian path**.
- $\mathrm{pe}(G) \leq 1 \Leftrightarrow G$ has a **dominating path**.

Note: P does not need to be induced.



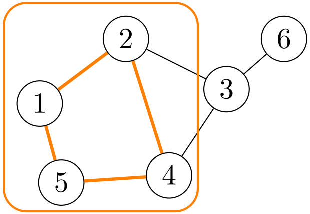
Spoiler: **Forbidding induced S_k and T_k if and only if a path of eccentricity $< k$ for every connected induced subgraph.**

Hereditary Graph Classes—Forbidden Induced Subgraphs

Definition (Hereditary graph class)

A class $\mathcal{C}$ is *hereditary* if it is **closed under taking induced subgraphs** (i.e., deleting vertices).

- Hereditary graph class can be characterized by a (potentially infinite) list of **minimal forbidden induced subgraphs**.



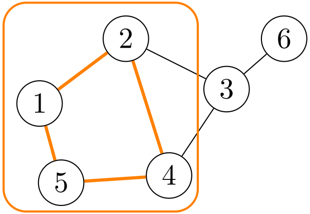
Hereditary Graph Classes—Forbidden Induced Subgraphs

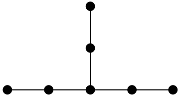
Definition (Hereditary graph class)

A class $\mathcal{C}$ is *hereditary* if it is **closed under taking induced subgraphs** (i.e., deleting vertices).

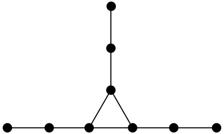
- Hereditary graph class can be characterized by a (potentially infinite) list of **minimal forbidden induced subgraphs**.

- S_k: a *subdivided claw*: all leaves at distance k from the center.
- T_k: the *line graph* of a subdivided claw.
- P_k: a *path* on k vertices.



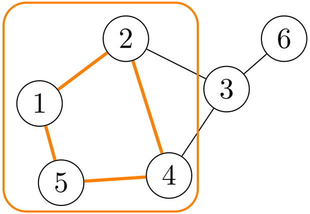


S_2



T_2

Hereditary Graph Classes—Forbidden Induced Subgraphs

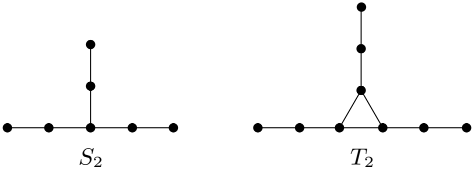


Definition (Hereditary graph class)

A class $\mathcal{C}$ is *hereditary* if it is **closed under taking induced subgraphs** (i.e., deleting vertices).

- Hereditary graph class can be characterized by a (potentially infinite) list of **minimal forbidden induced subgraphs**.

- S_k: a *subdivided claw*: all leaves at distance k from the center.
- T_k: the *line graph* of a subdivided claw.
- P_k: a *path* on k vertices.



Spoiler: **Forbidding induced S_k and T_k if and only if a path of eccentricity $< k$ for every connected induced subgraph.**

Motivation — Previous Results

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\mathrm{pe} = 0$).

Motivation — Previous Results

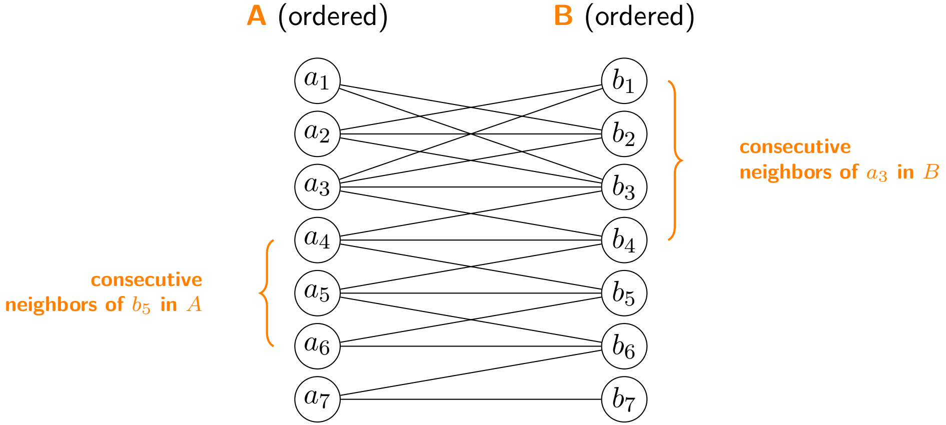
- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\mathrm{pe} = 0$).

Domination path examples ($\mathrm{pe} \leq 1$)

- **Bácsó & Tuza 1991**: In connected **P_5-free** graphs there is a **dominating clique or a dominating P_3**
- **Corneil, Olariu, and Stewart 1995-9**: In every connected **AT-free** graph there exists a **dominating pair** (s,t); in particular, a **shortest s-t path** is **dominating**.

Motivation — Previous Results

A graph G is **biconvex** if it is bipartite, with parts A and B that can each be linearly **ordered** so that for each vertex v of G, the neighborhood of v in the part not containing v forms a consecutive segment of vertices with respect to the linear ordering.



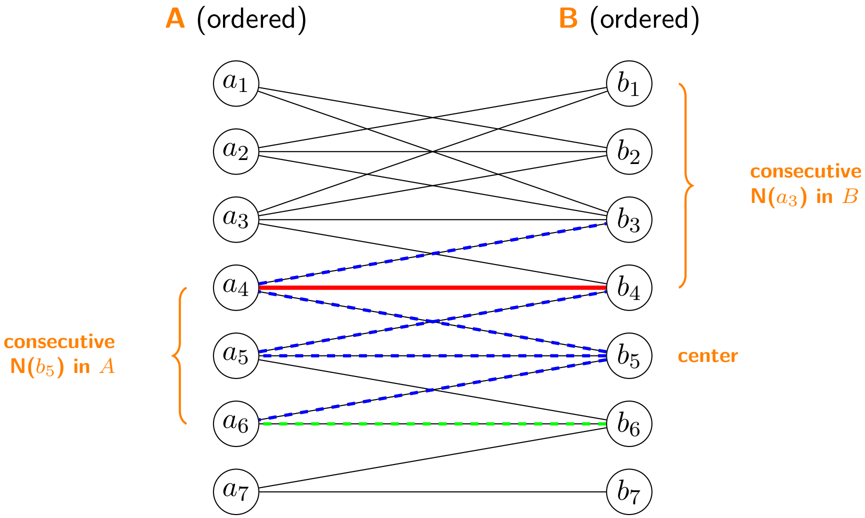
Motivation — Previous Results

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** $(\mathrm{pe} = 0)$.

Domination path examples $(\mathrm{pe} \leq 1)$

- **Bácsó & Tuza 1991**: In connected **P_5-free** graphs there is a **dominating clique or a dominating P_3**
- **Corneil, Olariu, and Stewart 1995-9**: In every connected **AT-free** graph there exists a **dominating pair** (s, t); in particular, a **shortest s-t path** is **dominating**.
- **Gómez & Gutiérrez 2023, Antony, Das, Gosavi, Jacob, Kulamarva 2024**: Every connected **biconvex** graph has a **dominating path** and there is a **linear-time** construction

Motivation — Previous Results



Motivation — Previous Results

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\mathrm{pe} = 0$).

Domination path examples ($\mathrm{pe} \leq 1$)

- **Bácsó & Tuza 1991**: In connected P_5**-free** graphs there is a **dominating clique or a dominating** P_3
- **Corneil, Olariu, and Stewart 1995-9**: In every connected **AT-free** graph there exists a **dominating pair** (s, t); in particular, a **shortest** s-t **path** is **dominating**.
- **Gómez & Gutiérrez 2023, Antony, Das, Gosavi, Jacob, Kulamarva 2024**: Every connected **biconvex** graph has a **dominating path** and there is a **linear-time** construction

Observe: **All the above graphs are** $\{S_2, T_2\}$**-free.**

Motivation — Previous Results

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\mathrm{pe} = 0$).

Domination path examples ($\mathrm{pe} \leq 1$)

- **Bácsó & Tuza 1991**: In connected P_5**-free** graphs there is a **dominating clique or a dominating** P_3
- **Corneil, Olariu, and Stewart 1995-9**: In every connected **AT-free** graph there exists a **dominating pair** (s, t); in particular, a **shortest** s-t **path** is **dominating**.
- **Gómez & Gutiérrez 2023, Antony, Das, Gosavi, Jacob, Kulamarva 2024**: Every connected **biconvex** graph has a **dominating path** and there is a **linear-time** construction

- **Bastide, Hilaire, and Robinson 2025**: graphs with **-C1P* satisfy $\mathrm{pe}(G) \leq 2$.

Motivation — Previous Results

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\mathrm{pe} = 0$).

Domination path examples ($\mathrm{pe} \leq 1$)

- **Bácsó & Tuza 1991**: In connected **P_5-free** graphs there is a **dominating clique or a dominating P_3**
- **Corneil, Olariu, and Stewart 1995-9**: In every connected **AT-free** graph there exists a **dominating pair** (s, t); in particular, a **shortest s-t path** is **dominating**.
- **Gómez & Gutiérrez 2023, Antony, Das, Gosavi, Jacob, Kulamarva 2024**: Every connected **biconvex** graph has a **dominating path** and there is a **linear-time** construction

- **Bastide, Hilaire, and Robinson 2025**: graphs with **-C1P* satisfy $\mathrm{pe}(G) \leq 2$.

It can be proved: **The above graphs are $\{S_2, T_1\}$-free.**

Main Theorem

Theorem (Our Main Theorem)

For every integer $k \geq 1$ and every graph G, the following statements are ***equivalent****:*

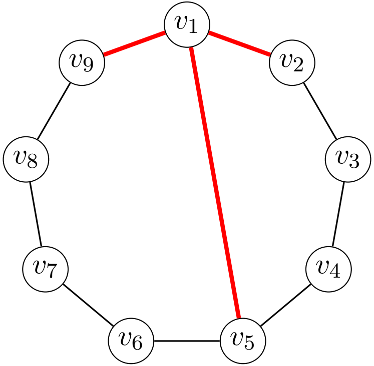
- *Every connected induced subgraph H of G satisfies $\mathrm{pe}(H) < k$.*
- *G is $\{S_k, T_k\}$**-free**.*

Main Theorem

Theorem (Our Main Theorem)

For every integer $k \geq 1$ and every graph G, the following statements are ***equivalent****:*

- ***Every connected induced*** *subgraph H of G satisfies $\mathrm{pe}(H) < k$.*
- *G is $\{S_k, T_k\}$****-free****.*



Main Theorem

Theorem (Our Main Theorem)

*For every integer $k \geq 1$ and every graph G, the following statements are **equivalent**:*

- *Every connected induced subgraph H of G satisfies $\mathrm{pe}(H) < k$.*
- *G is $\{S_k, T_k\}$-**free**.*

Consequences $\sim$ Recall: **Graphs below are $\{S_2, T_2\}$-free.**

- Bacsó & Tuza 1991: P_5-**free** graphs
- Corneil, Olariu, and Stewart 1995-9: **AT-free** graphs
- Gómez & Gutiérrez 2023, Antony, Das, Gosavi, Jacob, Kulamarva 2024: **biconvex** graphs

Main Thm $\Rightarrow$ Dominating path ($\mathrm{pe}(H) < 2$)

Main Theorem

Theorem (Our Main Theorem)

For every integer $k \geq 1$ and every graph G, the following statements are ***equivalent****:*

- *Every connected induced subgraph H of G satisfies $\mathrm{pe}(H) < k$.*
- *G is $\{S_k, T_k\}$-**free**.*

Consequences $\sim$ Recall: **Graphs below are even $\{S_2, T_1\}$-free.**

- Bastide, Hilaire, and Robinson 2025: graphs with **-C1P* satisfy $\mathrm{pe}(G) \leq 2$.

Improved: **MainThm $\Rightarrow$ Dominating path ($\mathrm{pe}(H) < 2$)**

Main Theorem

Theorem (Our Main Theorem)

For every integer $k \geq 1$ and every graph G, the following statements are ***equivalent****:*

- *Every connected induced subgraph H of G satisfies $\mathrm{pe}(H) < k$.*
- *G is $\{S_k, T_k\}$-**free**.*

Corollary (**Single** Forbidden Induced Subgraph)

Let H be a graph and $k \geq 1$. Then, the following statements are ***equivalent****:*

- *Every connected H-free graph G has $\mathrm{pe}(G) < k$.*
- *H is an induced subgraph of $3P_k$ **or** $P_{2k+1} + P_{k-1}$.*

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs.
Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\text{leaf}}$-free**.*

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class ***connected graphs that is hereditary and non-trivial*** *Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.*
Then, a graph G is ***hereditarily k-dominated by $\mathcal{D}$*** *if and only if G is* ***$\mathcal{F}_k$-free****.*

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{🍃}$-free**.*

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\text{leaf}}$-free**.*

A graph G is **hereditarily dominated by $\mathcal{D}$** if each connected induced subgraph of G contains a **dominating set** that induces a graph **from $\mathcal{D}$**.

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{🍃}$-free**.*

A graph G is hereditarily dominated by $\mathcal{D}$ if each connected induced subgraph of G contains a dominating set that induces a graph from $\mathcal{D}$.

$\mathcal{F}_k^{🍃}$ be the family of all k-leaf graphs of graphs in $\mathcal{F}$.

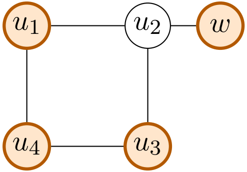
Main Tool: Bacsó & Tuza (2012)

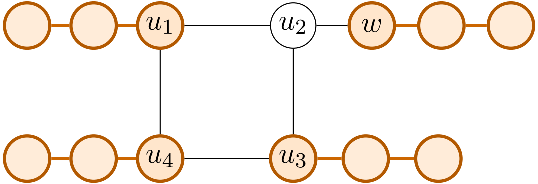
Theorem (Bacsó & Tuza 2012)

Then, a graph G is ***hereditarily k-dominated by $\mathcal{D}$*** *if and only if G is* ***$\mathcal{F}_k^{\text{🍃}}$-free****.*

A graph G is hereditarily dominated by $\mathcal{D}$ if each connected induced subgraph of G contains a dominating set that induces a graph from $\mathcal{D}$.

$\mathcal{F}_k^{\text{🍃}}$ be the family of all k-leaf graphs of graphs in $\mathcal{F}$.





Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\text{🍃}}$-free**.*

Set $\mathcal{D}$ to be the class of all connected **$\{S_1, T_1\}$-free graphs**.

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\varnothing}$-free**.*

Set $\mathcal{D}$ to be the class of all connected **$\{S_1, T_1\}$-free graphs**. Then $\mathcal{F}_{k-1}^{\varnothing} = \{S_k, T_k\}$.

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\varnothing}$-free**.*

Set $\mathcal{D}$ to be the class of all connected **$\{S_1, T_1\}$-free graphs**. Then $\mathcal{F}_{k-1}^{\varnothing} = \{S_k, T_k\}$.
Hence, Bacsó and Tuza gives us **an induced $k-1$ dominating graph $D \in \mathcal{D}$**.

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\emptyset}$-free**.*

Set $\mathcal{D}$ to be the class of all connected **$\{S_1, T_1\}$-free graphs**. Then $\mathcal{F}_{k-1}^{\emptyset} = \{S_k, T_k\}$.
Hence, Bacsó and Tuza gives us **an induced $k-1$ dominating graph $D \in \mathcal{D}$**. Which using

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\text{pe} = 0$).

yields a Hamiltonian path in D.

Main Tool: Bacsó & Tuza (2012)

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\text{leaf}}$-free**.*

Set $\mathcal{D}$ to be the class of all connected **$\{S_1, T_1\}$-free graphs**. Then $\mathcal{F}_{k-1}^{\text{leaf}} = \{S_k, T_k\}$.
Hence, Bacsó and Tuza gives us **an induced $k-1$ dominating graph $D \in \mathcal{D}$**. Which using

- **Duffus, Jacobson, and Gould 1981**: $\{S_1, T_1\}$-free graph has a **Hamiltonian path** ($\text{pe} = 0$).

yields a Hamiltonian path in D.
Hence, a **path of eccentricity $< k$** in any connected $\{S_k, T_k\}$-free graph.

Algorithmic Result

Note that the path eccentricity problem is **NP-complete** (**Müller 1996**)

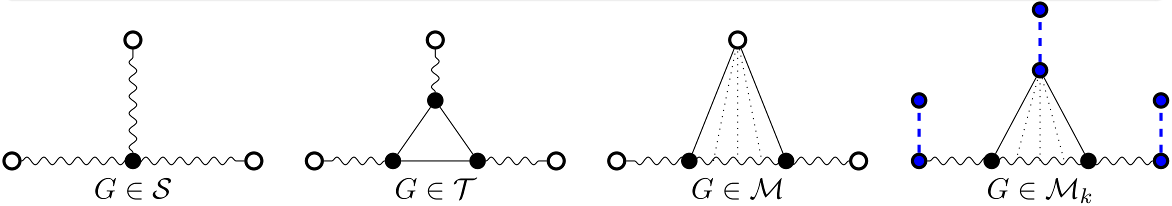
Algorithmic Result

Note that the path eccentricity problem is **NP-complete** (**Müller 1996**)

Theorem (Algorithmic Tool)

There is an ***algorithm*** *running in time* $\mathcal{O}(n^2(n+m))$ *that, given a connected graph* G *with* n *vertices and* m *edges, and an integer* $k \geq 1$*, finds one of the following:*

- *a path with* ***eccentricity*** $< k$ *in* G*, or*
- *an induced subgraph* H *of* G *isomorphic to either* S_k*,* T_k*, or a graph in* $\mathcal{M}_k$*.*

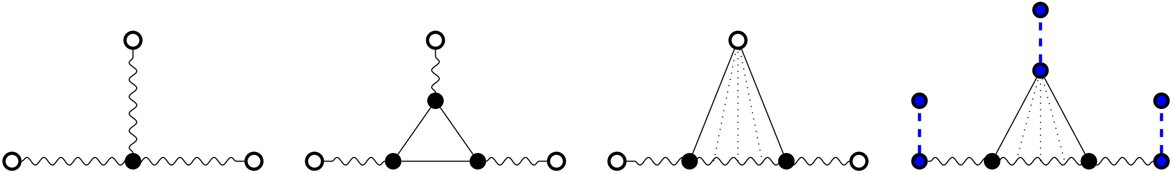


Algorithmic Result

Theorem (Simplified Algorithmic Tool)

There is an ***algorithm*** *running in time* $\mathcal{O}(n^2(n+m))$ *that, given a connected graph* G *with* n *vertices and* m *edges, and an integer* $k \geq 1$*, finds one of the following:*

- *a path with* ***eccentricity*** $< k$ *in* G*, or*
- *an induced subgraph* H *of* G *isomorphic to* S_k- ***leaf or*** T_k- ***leaf****.*



$G \in \mathcal{S}$ $G \in \mathcal{T}$ $G \in \mathcal{M}$ $G \in \mathcal{M}_k$

Algorithmic Result

Theorem (Simplified Algorithmic Tool)

There is an ***algorithm*** *running in time $\mathcal{O}(n^2(n+m))$ that, given a connected graph G with n vertices and m edges, and an integer $k \geq 1$, finds one of the following:*

- *a path with* ***eccentricity*** $< k$ *in G, or*
- *an induced subgraph H of G isomorphic to* S_k- ***leaf or*** T_k- ***leaf****.*

Conclusions & Open Questions

Open questions

- Algorithmic version of full main theorem.

That is, algorithmically in polynomial time find an **induced** S_k, or an **induced** T_k, or **a path of eccentricity** $< k$.

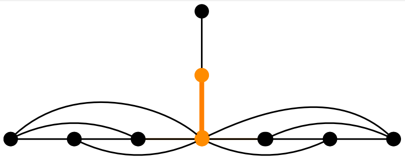
Conclusions & Open Questions

Open questions

- Algorithmic version of full main theorem.
- What are the graphs G such that every connected induced subgraph of G has a **longest** path that is **dominating**?

Theorem (Longest Dominating path)

*Every connected $3P_2$-free graph G has a **longest** path that is dominating.*



An example of a $\{P_5, 4P_2\}$-free graph no longest path of which is dominating.

Conclusions & Open Questions

Open questions

- Algorithmic version of full main theorem.
- What are the graphs G such that every connected induced subgraph of G has a **longest** path that is **dominating**?
- Algorithmic version of Bacsó & Tuza 2012.

Determine graph families $\mathcal{D}$ such that given G and $k \geq 1$, there is a polynomial-time algorithm finding an **induced graph** $D \in \mathcal{D}$ such that D k-dominates G or $F \in \mathcal{F}_k^{\text{🍃}}$.

Theorem (Bacsó & Tuza 2012)

Let $k \in \mathbb{N}$. Let $\mathcal{D}$ be a class of connected graphs that contains the class of all paths, excludes at least one connected graph, and is closed under taking connected induced subgraphs. ($\mathcal{D}$ hereditary and non-trivial) Let $\mathcal{F}$ be the family of all connected graphs F that do not belong to $\mathcal{D}$, but all proper connected induced subgraphs of F belong to $\mathcal{D}$.
*Then, a graph G is **hereditarily k-dominated by $\mathcal{D}$** if and only if G is **$\mathcal{F}_k^{\text{🍃}}$-free**.*

Conclusions & Open Questions

Open questions

- Algorithmic version of full main theorem.
- What are the graphs G such that every connected induced subgraph of G has a **longest** path that is **dominating**?
- Algorithmic version of Bacsó & Tuza 2012.

Thank you for your attention!