Proper Rainbow Saturation Numbers for Cycles

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C&C 2025

F-saturated graphs

A graph G is *F-saturated* if it is:

- **F-free** (i.e., does not contain F as a subgraph),
- but for **any** $e \in \binom{V(G)}{2} \setminus E(G)$, the graph $G + e$ contains F.

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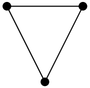

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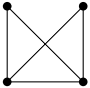

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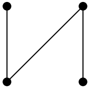

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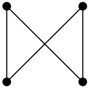

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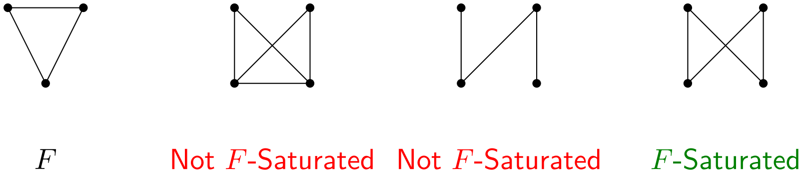

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Question: given F and n, how sparse (or dense) can an n-vertex, F-saturated graph be?

Turán and Saturation Numbers

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Classical problems with very extensive history.

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Given an integer t and a graph H. The *t-rainbow saturation number* is the **minimum number of edges** in a t-edge-colored graph G on n vertices such that G **does not contain a rainbow copy of H**, but adding to G a **new edge in any color** creates a **rainbow copy of H**.

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Does not assume a setting of proper edge-colorings.

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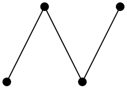

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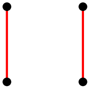

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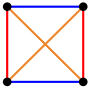

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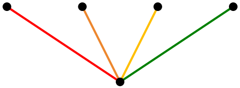

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Largest Cases

The **maximum number of edges** in an n-vertex, properly rainbow F-saturated graph is the *rainbow Turán number* $\mathrm{ex}^*(n, F)$.

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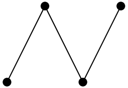

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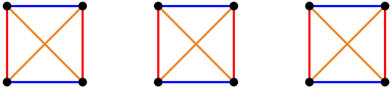

$\mathrm{ex}^*(n, P_3) \approx \frac{3n}{2}$

Comparison: $\mathrm{ex}(n, P_3) \approx n$.

Smallest Cases

The **minimum number of edges** in an n-vertex, properly rainbow F-saturated graph is the *(proper) rainbow saturation number* $\mathrm{sat}^*(n, F)$.

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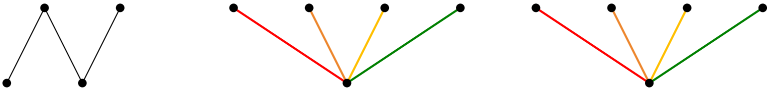

$F = P_3$ $\qquad$ $\mathrm{sat}^*(n, P_3) \approx \frac{4n}{5}$

Comparison: $\mathrm{sat}(n, P_3) \approx \frac{n}{2}$.

Known Values of $\mathrm{sat}^*(n, F)$

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Very rough proof idea: disjoint copies of $K_{1,4}$ gives the upper bound. For the lower bound, what components of a rainbow saturated graph could be sparser than $K_{1,4}$?

Known Values of $\mathrm{sat}^*(n, F)$

Theorem (HLM 2025)

For all n,

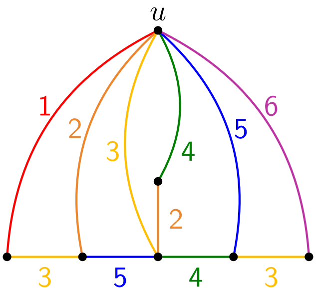
$$\mathrm{sat}^*(n, C_4) \leq \frac{11n}{6} + O(1).$$

Moreover, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, if $n \geq n_0$ and G is an n-vertex, properly rainbow C_4-saturated graph, then G has more than $\left(\frac{11}{6} - \varepsilon\right) n$ edges.

Previously known: Bushaw, Johnston, and Rombach bounded

$$n \leq \mathrm{sat}^*(n, C_4) \leq 2n - 2.$$

Determining $\mathrm{sat}^*(n, C_4)$: A Construction



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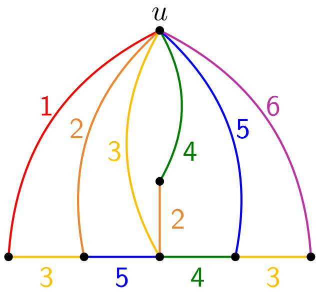

To scale up this construction: Take a universal u adjacent to coppies of $S_{2,2,1}$.

Number of edges: $n-1$ edges ending at u. For the rest, sets of 6 vertices yield 5 edges each. Total is $\approx \frac{11n}{6}$.

Properties and Intuition Helping with the Upper-bound Analysis

Lemma

Let G be a graph and $v \in V(G)$. If there exists a proper edge-coloring of G which is rainbow C_4-free, then the subgraph of G induced on $N(v)$ ***does not contain*** *the following subgraphs:*

1. *A copy of K_3 with pendant edges from two vertices;* 
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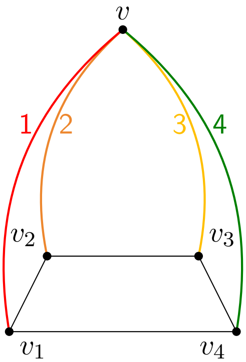

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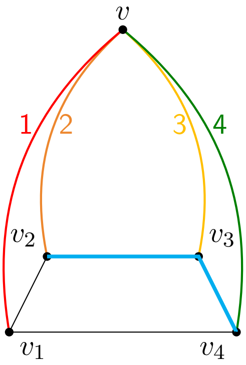

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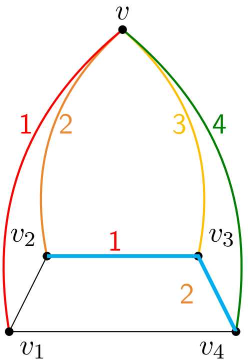

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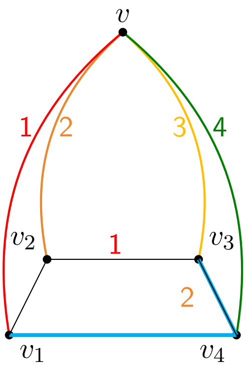

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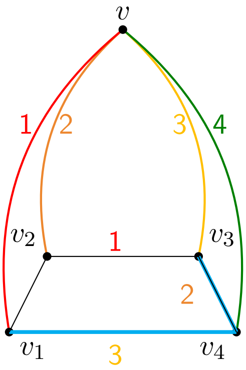

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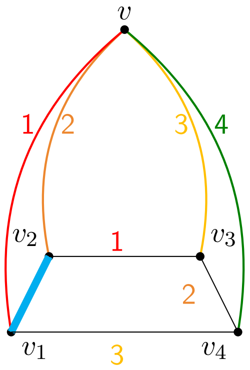
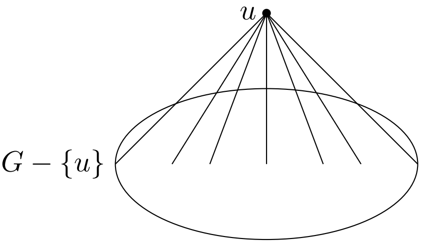
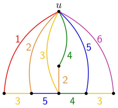
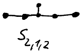


Lower Bound Ideas

Suppose first we have a universal vertex u.







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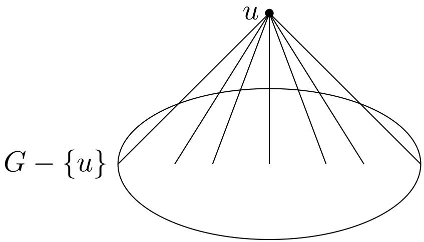

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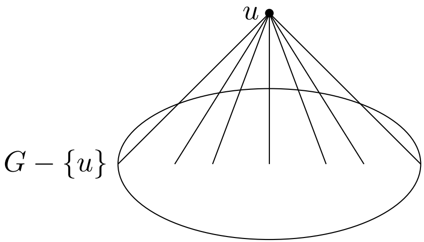

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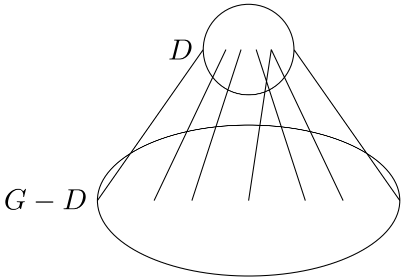
Problem: it is not at all clear that G has a universal vertex.

Dominating Sets

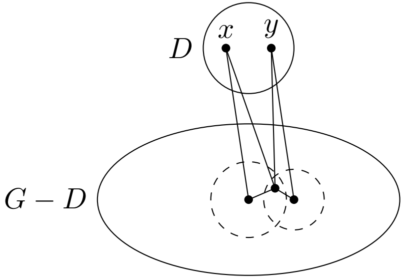
A **dominating set** in a graph G is a set D of vertices such that every vertex of $V(G) \setminus D$ is adjacent to something in D.



New Idea: A nice dominating set might work sort of like a universal vertex.

Dominating Set Wrinkles

A nice dominating set is harder to work with than a universal vertex:



Problems: **too-sparse components in $G - D$**.

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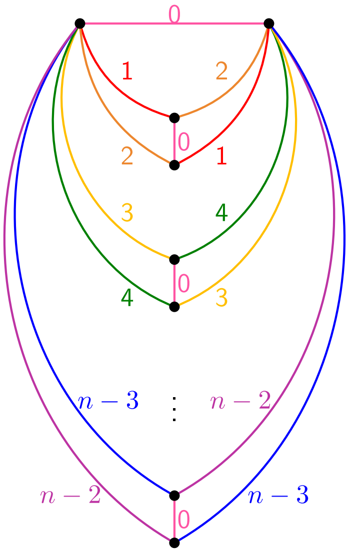
$\rightsquigarrow$ **Conclusion:** constant number of too-sparse components!

Longer Cycles

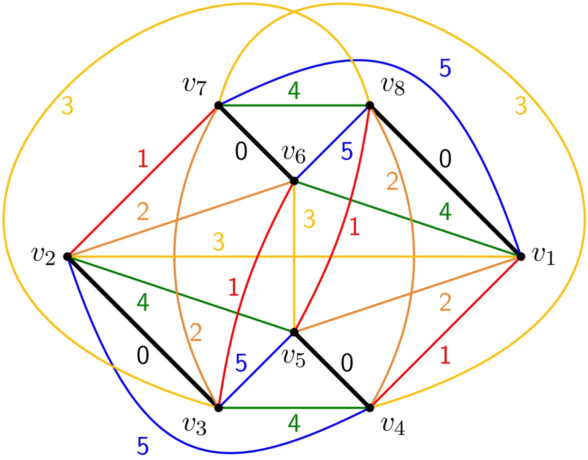
What about cycles on more than 4 edges?

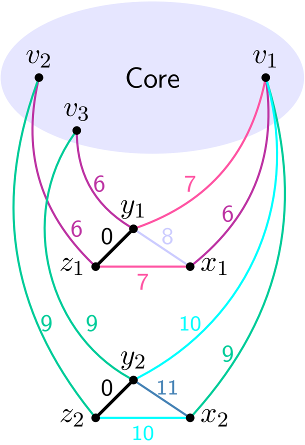
While some ideas from our proof may help, longer cycles seem to behave differently. In particular, the "u + trees" model is tough to extend.

We do offer a construction giving an upper bound of $\approx \frac{5n}{2}$ for $C_5 \ldots$



. . . and a construction giving $\approx \frac{7n}{3}$ for C_6!





Follow-up Developement Appearing in 2024

- **Paths.** $\mathrm{sat}^*(n, P_\ell) = n + O(1)$ (tight up to an additive constant).
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and a worse bound for C_8 of $5n - 12$.
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- **Trees.** Broad asymptotic results: if $\mathrm{diam}(T) \geq 5$, then $\mathrm{sat}^*(n, T) \geq n - 1$ (tight for several infinite families, e.g., brooms), giving $\Theta(n)$ for all connected trees with large diameter.
Lane–Morrison

Cycles: Current Best Bounds Summary

- C_4. $\text{sat}^*(n, C_4) = \frac{11}{6}n \pm o(n)$
- C_5. $\text{sat}^*(n, C_5) \leq \lfloor \frac{5}{2}n \rfloor - 4$.
- C_6. $\text{sat}^*(n, C_6) \leq \frac{7}{3}n + O(1)$.
- C_k **for** $k \geq 7$. $\text{sat}^*(n, C_k) \leq \frac{k-1}{2}n + O(1)$; except $\text{sat}^*(n, C_8) \leq 5n - 12$.

Open Questions

While we now know many more values of $\mathrm{sat}^*(n, F)$ than we did a year ago, many natural graphs remain unresolved.

- C_ℓ, for $\ell > 4$?

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Some nice general questions are also open:

- Is $\mathrm{sat}^*(n, F)$ **always larger than** $\mathrm{sat}(n, F)$? In all known cases, this is true (and in fact, there is a multiplicative factor of difference between the two).

Thanks for your attention!

Questions?