Random Embeddings of Graphs: The Expected Number of Faces in Most Graphs is Logarithmic

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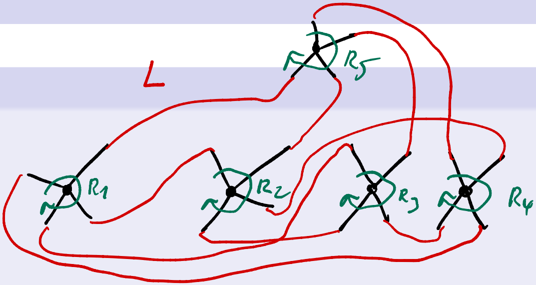
Combinatorial Embedding



Combinatorial maps

a triple $M = (D, R, L)$ where

- D is an abstract set of **darts**;
- R is a collection of **unicyclic permutations** on D;
- L is a fixed point free **involution** on D.

Combinatorial maps are in bijective correspondence with **2-cell embeddings** of graphs on **oriented** surfaces (up to orientation-preserving homeomorphisms).

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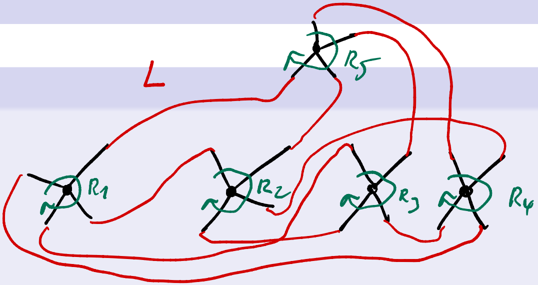

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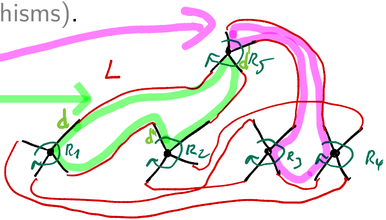
Faces $R \circ L$



Note that # Faces determines the genus by Euler's Formula.

Related results

Complete Graph K_n

The expected number of faces in a random embedding of the complete graph K_n is at most:

- Stahl 1995 $n + \ln n$
- Conjecture Mauk and Stahl 1996 $2 \ln n + O(1)$

General Graphs

The expected number of faces in a random embedding of any graph is at most:

- Stahl 1991 $n \ln n$
- Campion Loth and Mohar 2023 $\frac{\pi^2}{6} n$

Our Results—Comple Graphs

Complete Graph K_n

$H_n := \sum_{i=1}^{n} \frac{1}{i}$

The expected number of faces in a random embedding of the complete graph K_n is:

- Stahl 1995 $\leq n + \ln n$
- Conjecture Mauk and Stahl 1996 $\leq 2\ln n + O(1)$
- **CHMMŠ 2024** $\leq H_{n-3}H_{n-2}$

$\leq 10\ln n + 2$

(n large enough) $\leq 3.65\ln n$

$\geq 0.5\ln n - 2$

Our Results—Random Graphs

General Graphs

The expected number of faces in a random embedding of any graph is $\leq$

- Stahl 1991 $n \ln n$
- Campion Loth and Mohar 2023 $\frac{\pi^2}{6} n$

Theorem (Random Graphs)

Let $n \in \mathbb{N}$ and $p \in (0,1]$ $(p = p(n))$. Then the number of faces in a random embedding of a ***random graph in*** $G(n,p)$ *is* $\leq H_n^2 + 1/p$.

Theorem (Random Multigraphs)

Let $\mathbf{d} = (t_1, t_2, \ldots, t_n)$ be a degree sequence for an n-vertex multigraph (possibly with loops) where $t_i \geq 2$ for all i. Let $\mathbb{E}[F_{\mathbf{d}}]$ be the average number of faces in a random embedding of a ***random multigraph with degree sequence*** d. *Then* $\mathbb{E}[F_{\mathbf{d}}] = \Theta(\ln n)$.

Our Results—Random Graphs

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Theorem (Random Graphs of Bounded degree)

Let $d \geq 2$ be a constant, $\varepsilon > 0$, and let $\mathbf{d} = (t_1, t_2, \ldots, t_n)$ be a degree sequence for some n-vertex ***simple*** *graph with $2 \leq t_i \leq d$ for all i, and such that $m_{\mathbf{d}} \geq (1+\varepsilon)n$. Let $\mathbb{E}[F^s_{\mathbf{d}}]$ be the average number of faces in a random embedding of a random simple graph with degree sequence $\mathbf{d}$. Then $\mathbb{E}[F^s_{\mathbf{d}}] = \Theta_{\varepsilon,d}(\ln n)$ (constants within Θ depend on ε and d).*

Our Results—Random Graphs

Theorem (Random Multigraphs)

Let $\mathbf{d} = (t_1, t_2, \ldots, t_n)$ *be a degree sequence for an* n*-vertex multigraph (possibly with loops) where* $t_i \geq 2$ *for all* i*. Let* $\mathbb{E}[F_\mathbf{d}]$ *be the average number of faces in a random embedding of a* ***random multigraph with degree sequence*** d*. Then* $\mathbb{E}[F_\mathbf{d}] = \Theta(\ln n)$*.*

Corollary (Random Multigraphs)

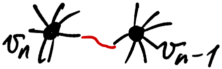
Let G *be a* ***random multigraph*** *with degree sequence* $\mathbf{d}$*. Then the probability that the number of faces in a random embedding of* G *is* $\geq c(\log(n) + 1)$ *is* $\leq \frac{4}{c}$*.*

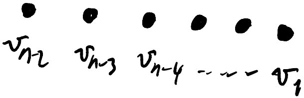
$\ln^2 n$ for K_n proof—The Random Process

Complete Graph K_n

The expected number of faces in a random embedding of the complete graph K_n is:
$\leq H_{n-3}H_{n-2}$.

- Process the vertices $v_n, \ldots, v_1$ in order.
- Start with v_n and v_{n-1}.





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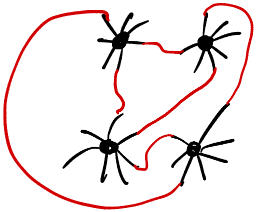
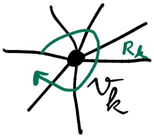
- Consider vertex v_k for $k \in [n-2]$.
  Label the darts of D_k as $\{d_1, \ldots, d_{n-1}\}$ arbitrarily.
  We define R_k as this cyclic order.





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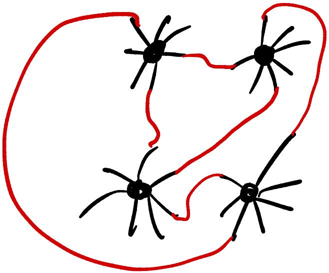

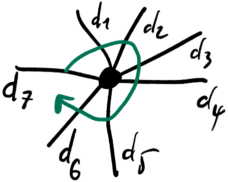

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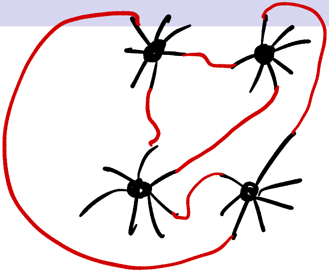

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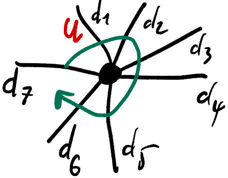
  - Process darts in D_k in order $d_1, d_2, \dots, d_{n-1}$.
    If $k > 1$, give d_1 the label u, remove one copy of u from C_k, and proceed processing d_2.
    If $k = 1$, start by processing d_1.



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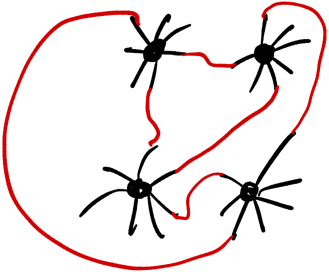

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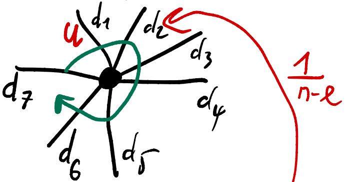

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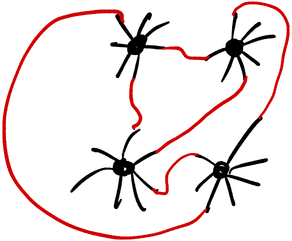

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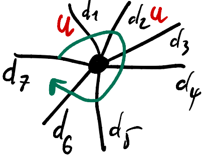
No Face is Finished



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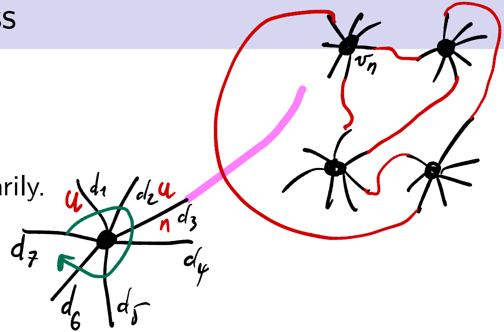
$\ln^2 n$ for K_n proof—The Random Process



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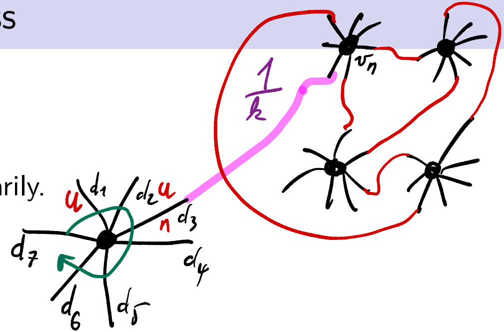

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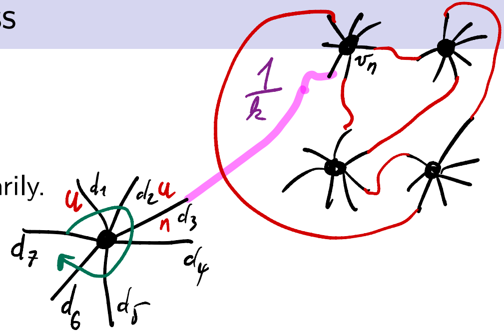

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We didn't finish a face but we could.

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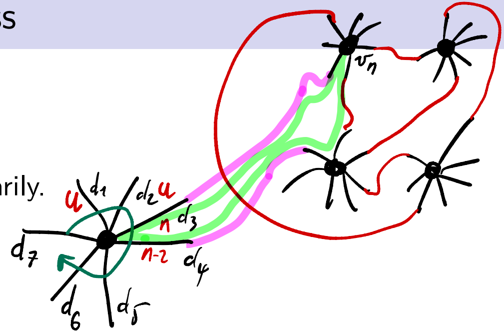

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For example now. But at most one choice can do it.

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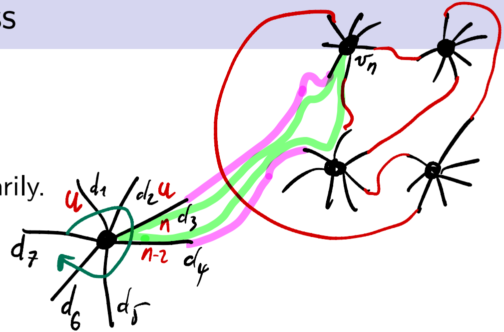
$\ln^2 n$ for K_n proof—The Random Process



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The probability is:

The previous dart is paired.

$$E[F] = E[F_1] + \sum_{k=2}^{n-2} \sum_{\ell=3}^{n-1} \frac{n-k}{n-2} \cdot \frac{1}{k(n-\ell)} < H_{n-3} H_{n-2}$$

$$C_k := \{\cancel{n}, n-1, \cancel{n-2}, n-3, \cancel{u}, \cancel{u}, u\}$$

$\ln n$ Theorem Main Obstacles

- More complicated random process (crafting both R and L),
- 1-open faces complication,

Options for Random embedding: 1) Fix R and generate all L ← the $\ln^2 n$ proof

2) Fix L and generate R ← degree sequence random gr

3) generate all L and R ← $\ln n$ proof

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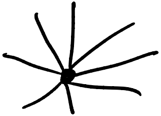
Sort darts around v_k



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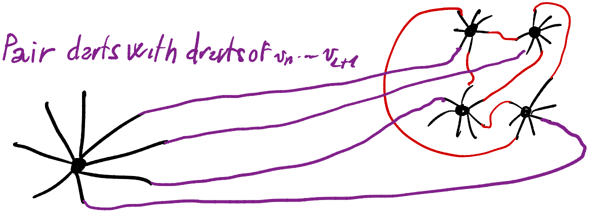
Sort darts around v_k ; Pair darts with darts of $v_n \cdots v_{k+1}$



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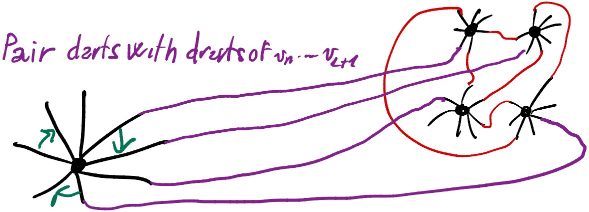
Sort darts around v_k ; Pair darts with darts of $v_n \cdots v_{k+1}$

determine R_k for d_ℓ

define $R_k(d_\ell)$



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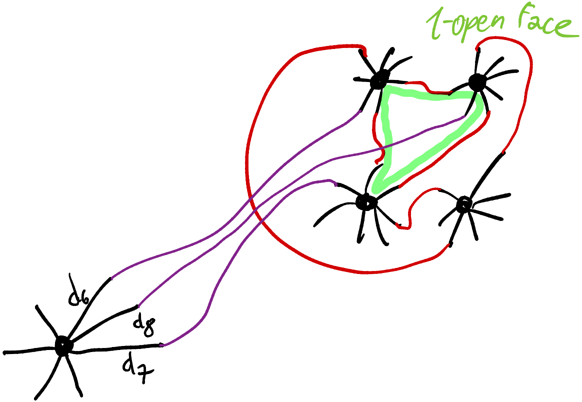

$\ln n$ Theorem Main Obstacles

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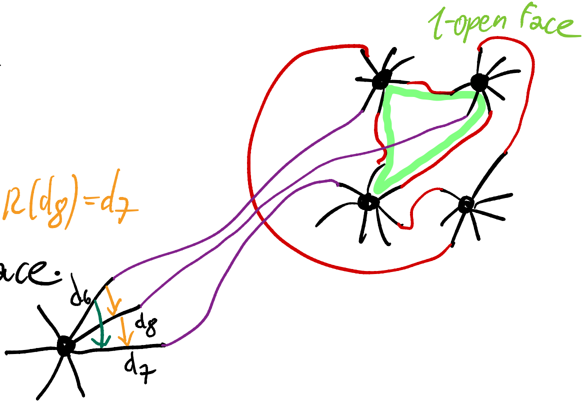
Two choices

$R(d_6) = d_7$

$R(d_6) = d_8$ and $R(d_8) = d_7$

Complete a face.



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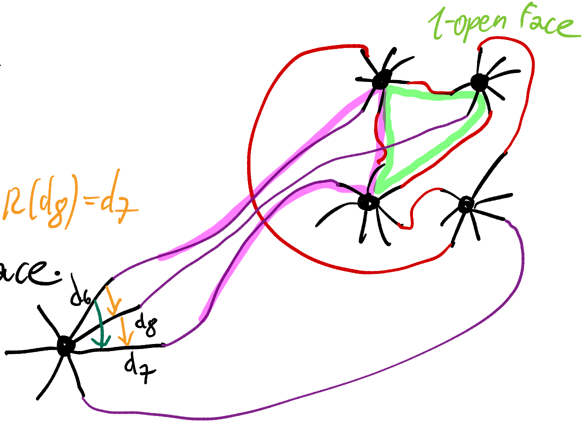
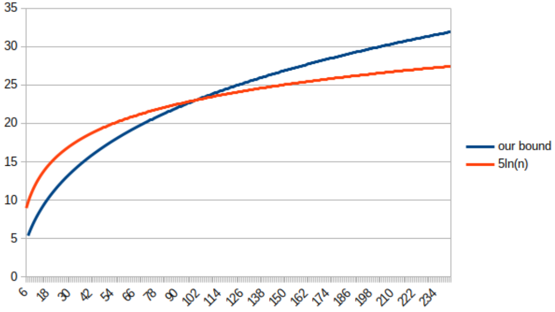
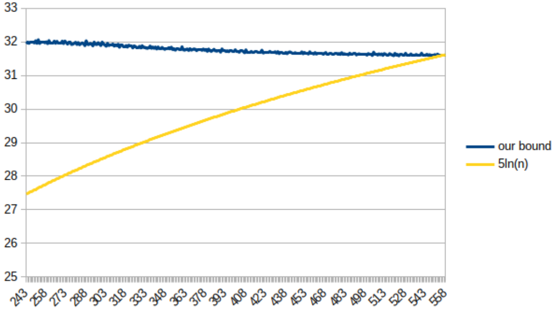


- More complicated random process (crafting both R and L),
- 1-open faces complication,
- Complicated analysis of the derived expression,
- Computer assisted evaluation for small n.





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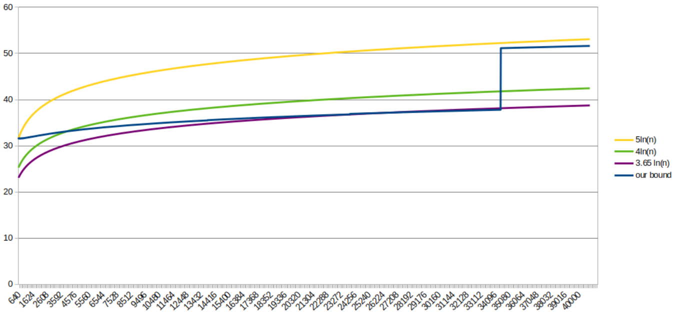

Open Problems

Conjecture (Mauk and Stahl 1996)

The expected number of faces in a random embedding of the complete graph K_n is $2\ln n + O(1)$.

Conjecture

The expected number of faces in a random embedding of a ***random graph*** $G \in G(n, M)$ *is* $(1 + o(1))\ln(M)$.

Conjecture

Let G be a graph on n vertices with ***minimum vertex degree*** $\Omega(n)$. *Then G satisfies* $\mathbb{E}[F] = \Theta(\ln(n))$.

Conjecture

The expected number of faces in a ***non-orientable*** *random embedding of the* ***complete graph*** K_n *is at most* $\ln(n) + O(1)$.

Open Problems

Conjecture

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Thank you!