Parameterized Approximation Schemes for Steiner Trees with Small Number of Steiner Vertices

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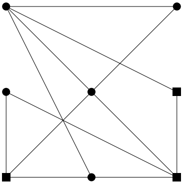
STACS 2018,
Caen, France

Steiner Tree

Steiner Tree
Input: A graph $G = (V, E)$, a set of terminals $R \subseteq V$.
Task: Find a tree $T \subseteq E$ of smallest size such that (R, T) is connected.

A vertex in $V \setminus R$ is called a Steiner vertex.
The number of Steiner vertices in the optimal solution is denoted by p, (i.e. $|V(T^*) \setminus R|$ for the optimum T^*).



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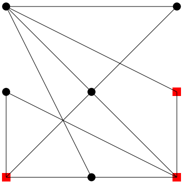

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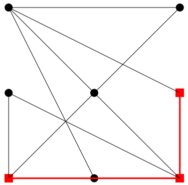

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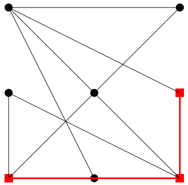

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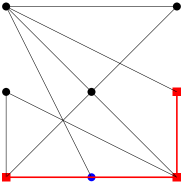

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Weighted Steiner Tree
Input: A graph $G = (V, E)$, a set of terminals $R \subseteq V$, and a weight function $w\colon E \to \mathbb{R}^+$.
Task: Find a tree $T \subseteq E$ of smallest weight such that (R, T) is connected.

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Directed Steiner Tree
Input: A directed graph $G = (V, A)$, a set of terminals $R \subseteq V$, a root $r \in R$.
Task: Find an arborescence $T \subseteq A$ of smallest size such that every terminal is reachable from r by edges in T.

Steiner Tree and Parameters

Two natural parameters of the problem.

- the number of terminals (denoted by $|R|$),
- the number of Steiner vertices in the optimal solution (denoted by p) i.e. $|V(T^*) \setminus R|$.

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Note, that if the number of Steiner vertices is considered as a parameter instead there is a trivial FPT algorithm.

Steiner Tree – Known Results

Parameterized complexity:

- FPT w.r.t. number of terminals
  - $3^k n^3$ algorithm in the 70s [Dreyfus & Wagner 71],
  - the best known now is $2^k n^2$ [Björklund et al. 07] for the unweighted case.
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Approximation:

- 2-approximation is simple
- $(\ln 4 + \varepsilon)$-approximation algorithm [Byrka et al. 13]
- (96/95)-approximation is NP-hard (which means no approximation scheme exists unless $\mathsf{P} = \mathsf{NP}$) [Chlebík and Chlebíková 02]

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Since Steiner Tree is both hard to approximate and W[1]-hard w.r.t. p, it is an excellent target for parameterized approximation.

Steiner Tree – Our Results

Theorem

For any computable functions g, f, it is impossible to compute an $f(p)$-approximation for the Weighted Directed Steiner Tree problem parameterized by p in time $g(p) \cdot n^{O(1)}$, unless $\mathsf{W}[1] = \mathsf{FPT}$.

Results	Undirected	Directed
Unweighted		
Weighted		$f(p)$-approx. hard

Steiner Tree – Our Results

Definition (EPAS)

Efficient Parameterized Approximation Scheme for minimization problem is defined as follows: For all $\varepsilon > 0$ there exists an algorithm that runs in time $f(\varepsilon, p) \cdot n^{O(1)}$ and returns $(1 + \varepsilon)$-approximation.

Steiner Tree – Our Results

Theorem

There is an EPAS for the Unweighted Directed Steiner Tree problem

in time $2^{p^2/\varepsilon} \cdot n^{O(1)}$.

Results	Undirected	Directed
Unweighted		EPAS
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Steiner Tree – Our Results

Theorem

There is an EPAS for the Weighted Undirected Steiner Tree problem

in time $2^{O(p^2/\varepsilon^4)} \cdot n^{O(1)}$.

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- If not then we have a bounded number of terminals in terms of ε and p: $|R| \leq \frac{2p}{\varepsilon}$.
- So, we can use a known algorithm parameterized by $|R|$.

Steiner Tree – Our Results

Theorem

There is an EPAS for the Weighted Undirected Steiner Tree problem in time $2^{O(p^2/\varepsilon^4)} \cdot n^{O(1)}$.

Results	Undirected	Directed
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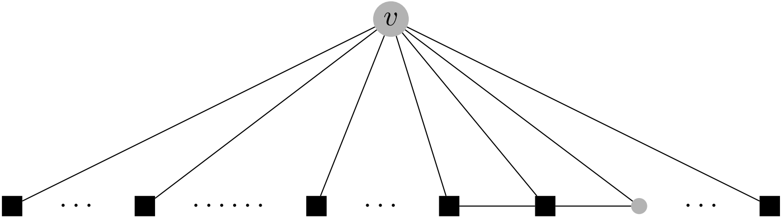
Weighted	EPAS	$f(p)$-approx. hard

Overview of algorithm

We define a star C as the central vertex v and the subset Q of its adjacent terminals, where $|Q| \geq 2$.

We define a ratio of a star C, as

$$\frac{w(C)}{(|Q|-1)}.$$



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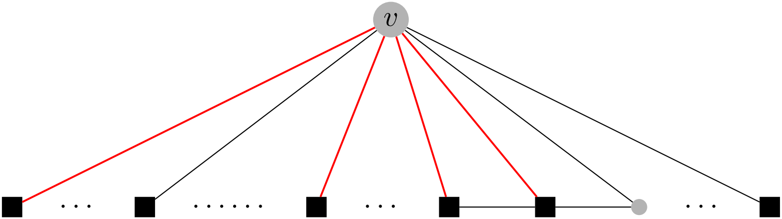

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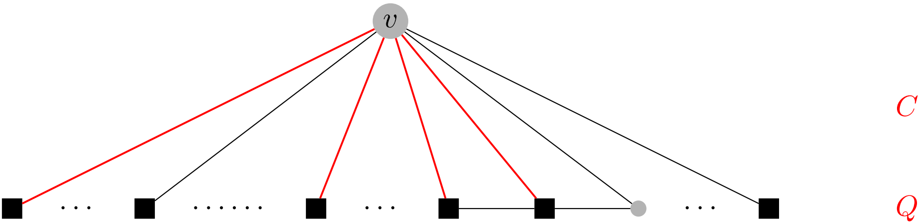

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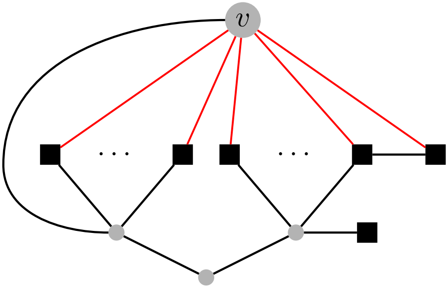

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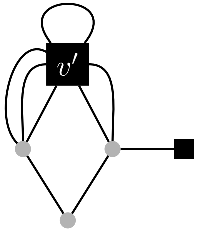

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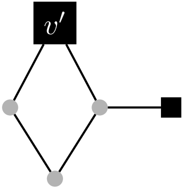

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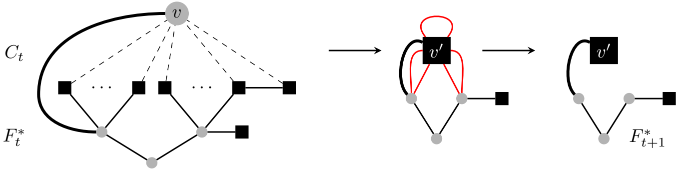
- We maintain a tree originating from T^* in the contracted instances.
- Denote the tree after t steps of the algorithm by T_t^*.



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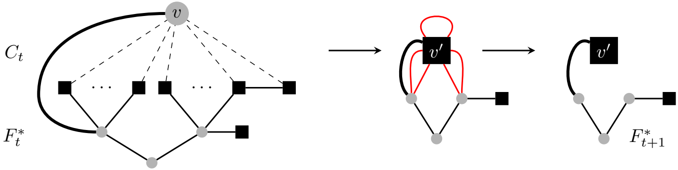

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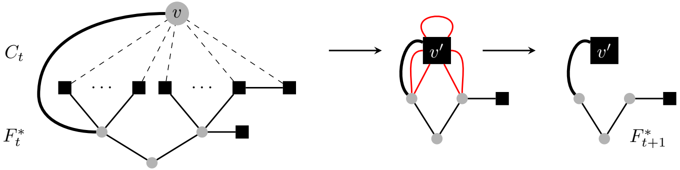

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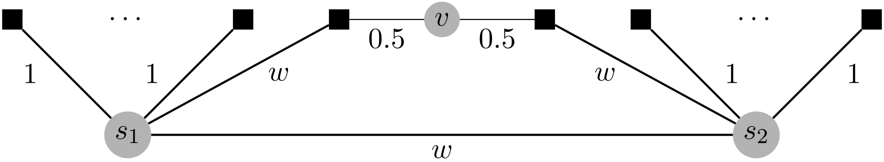

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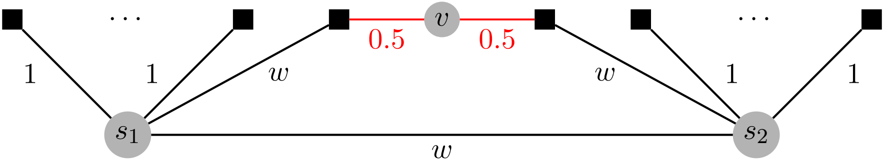

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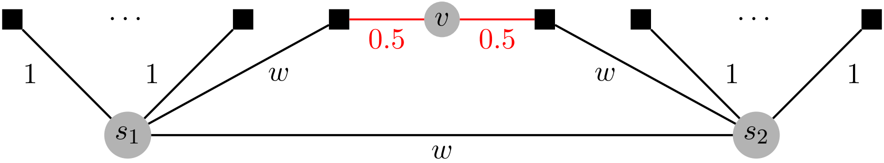

Proposition

If a star C contains at least $(1+\varepsilon)p/\varepsilon$ terminals then contraction of C is good.

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- We rescale weights to be ≥ 1.

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- otherwise to prove that the number of terminals is bounded in therms of p and ε.

Kernelization – Our Results

Definition (PSAKS)

Polynomial Size Approximate Kernelization Scheme for minimization problem is defined as follows: For all $\varepsilon > 0$ there exists an algorithm that runs in polynomial time ($n^{\mathrm{O}(1)}$) s.t. it returns an $(1+\varepsilon)$-approximate kernel of size $\mathrm{O}(p^{f(\varepsilon)})$.

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There is a PSAKS for the Weighted Undirected Steiner Tree problem. It computes an $(1+\varepsilon)$-approximate kernel

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- Combining these two we get a polynomial size $(1+\varepsilon)$-approximate kernel

  of size$(p/\varepsilon)^{2^{\mathcal{O}(1/\varepsilon)}}$.

Generalizations of Steiner Tree

Weighted Steiner Forest

Input: A graph $G = (V, E)$, a list of terminal pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$, and a weight function $w\colon E \to \mathbb{R}^+$.

Task: Find a forest $F \subseteq E$ of smallest weight such that s_i and t_i is in the same component of F for every i.

We extend our results to Steiner Forest while parameterized by the number of components of the optimal solution in addition to p.

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