Single-conflict colorings of degenerate graphs

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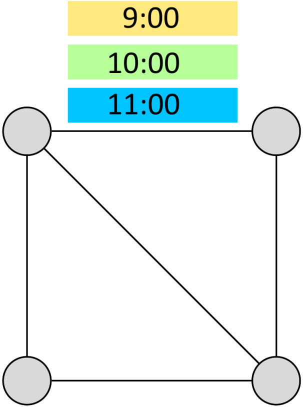
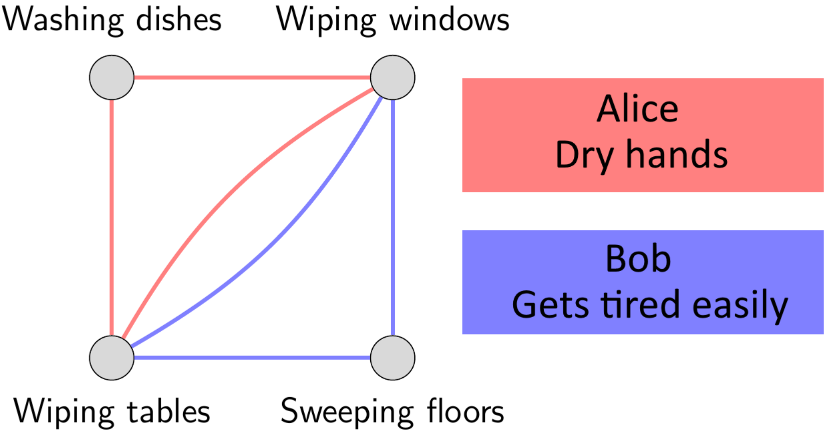
Scheduling problems

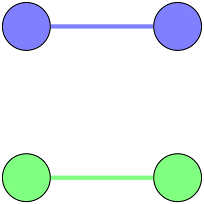


Job assignments



Adapted graph coloring: definition

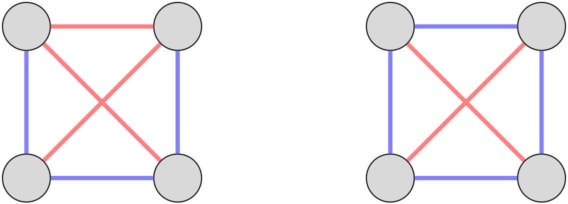
- A coloring of $E(G)$ with $\{1, \ldots, k\}$ is given.
- Goal is to color $V(G)$ using $\{1, \ldots, k\}$ without monochromatic edges.



(Hell, Zhu 2008)

Adaptable chromatic number (χ_{ad})

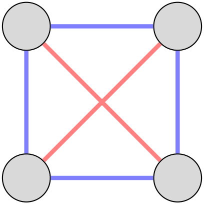
- Does G *always* have an adapted coloring when using k colors?



- Sometimes 2 colors are not enough for K_4.

Adaptable chromatic number (χ_{ad})

- If G *always* has an adapted coloring using $\{1, \dots, k\}$, then $\chi_{ad}(G) \le k$.



- $\chi_{ad}(K_4) \ge 3$.

Bounds on χ_{ad}

- To show $\chi_{ad}(G) \leq k$, we must show that G has an adapted coloring for *every* edge-coloring using $\{1, \ldots, k\}$.
- $\chi_{ad}(G) \leq \chi(G)$.
- ???

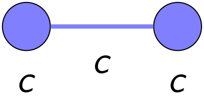
Bounds on χ_{ad}

- To show $\chi_{ad}(G) \leq k$, we must show that G has an adapted coloring for *every* edge-coloring using $\{1, \ldots, k\}$.
- $\chi_{ad}(G) \leq \chi(G)$.
- Bounds using the *probabilistic method*: We randomly color G, and we show that if we are lucky, then we get an adapted coloring.

Bad events

- Coloring of $E(G)$ with $\{1, \ldots, k\}$ is given.
- We randomly color $V(G)$ with $\{1, \ldots, k\}$.

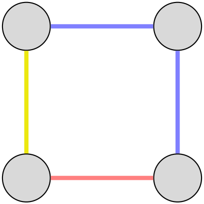
Bad event:



- A bad event happens with probability $\frac{1}{k^2}$.
- If no bad event occurs, then we get an adapted coloring.

Dependent bad events

- Two bad events are *dependent* if and only if they share a vertex.



- A bad event at e is dependent with $\deg(e)$ other bad events.

Lovász Local Lemma

- A set of bad events is given.
- Each bad event dependent with $< D$ other bad events.
- Each bad event has probability $\leq p$.

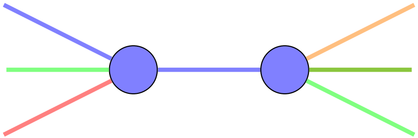
If

$$epD \leq 1,$$

then with positive probability, no bad event occurs.

Bounds on χ_{ad}

- Consider G of max deg Δ edge-colored with $\{1, \ldots, k\}$.



- $p = \frac{1}{k^2}$, $D = 2\Delta$
- $\chi_{ad}(G) \leq \left\lceil \sqrt{2e\Delta} \right\rceil$.

χ and χ_{ad}

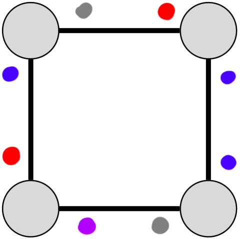
- $\chi_{ad}(G) \geq (1 + o(1))\sqrt{\chi(G)}$. (Molloy, 2017)
- Often $\sqrt{\chi(G)}$ and $\chi_{ad}(G)$ are $\Theta(\sqrt{\Delta})$.
- For girth 5, $\sqrt{\chi(G)}$ and $\chi_{ad}(G)$ are $\Theta\left(\sqrt{\frac{\Delta}{\log \Delta}}\right)$ (Aliaj, Molloy 2021)

Question

When else does $\chi_{ad}(G) \approx \sqrt{\chi(G)}$ hold?

Single-conflict colorings

- A *conflict* from $\{1, \dots, k\}^2$ is given to each edge.
- Goal is to color G while avoiding conflicts.



- Monochromatic conflicts $\implies$ adapted coloring

(Dvořák, Esperet, Kang, Ozeki 2018)

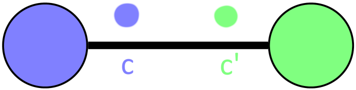
Single-conflict chromatic number ($\chi_{\not\leftrightarrow}$)

- If G *always* has an single-conflict coloring using $\{1, \ldots, k\}$, then $\chi_{\not\leftrightarrow}(G) \leq k$.
- $\chi_{\not\leftrightarrow}(G) \leq \Delta + 1$
- $\chi_{\not\leftrightarrow}(G) \leq \text{degeneracy}(G) + 1$

Bad events

- Conflicts on $E(G)$ using $\{1, \dots, k\}$ are given.
- We randomly color $V(G)$ with $\{1, \dots, k\}$.

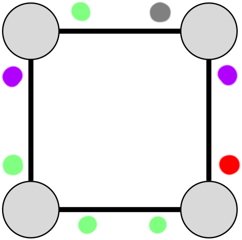
Bad event:



- A bad event happens with probability $\frac{1}{k^2}$.
- If no bad event occurs, then we get a single-conflict coloring.

Dependent bad events

- Two bad events are *dependent* if and only if they share a vertex.



- A bad event at e is dependent with $\deg(e)$ other bad events.

Lovász Local Lemma

- A set of bad events is given.
- Each bad event dependent with $< D$ other bad events.
- Each bad event has probability $\leq p$.

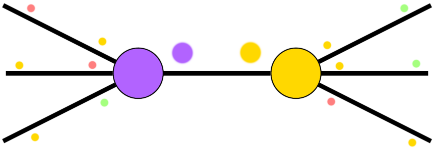
If

$$epD \leq 1,$$

then with positive probability, no bad event occurs.

Single-conflict chromatic number ($\chi_{\not\leftrightarrow}$)

- Consider G of max deg Δ with conflicts from $\{1, \ldots, k\}$.



- $p = \frac{1}{k^2}$, $D = 2\Delta$
- $\chi_{\not\leftrightarrow}(G) \leq \left\lceil \sqrt{2e\Delta} \right\rceil$.

χ and $\chi_{\leftrightarrow}$

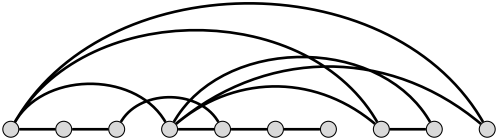
- $\chi_{\leftrightarrow}(G) \geq \chi_{ad}(G) \geq (1+o(1))\sqrt{\chi(G)}$. (Molloy, 2017)
- Often $\sqrt{\chi(G)}$ and $\chi_{\leftrightarrow}(G)$ are $\Theta(\sqrt{\Delta})$.
- For girth 5, $\sqrt{\chi(G)}$ and $\chi_{\leftrightarrow}(G)$ are $\Theta\left(\sqrt{\frac{\Delta}{\log \Delta}}\right)$ (Aliaj, Molloy 2021)

Question

When else does $\chi_{\leftrightarrow}(G) \approx \sqrt{\chi(G)}$ hold?

Graph degeneracy

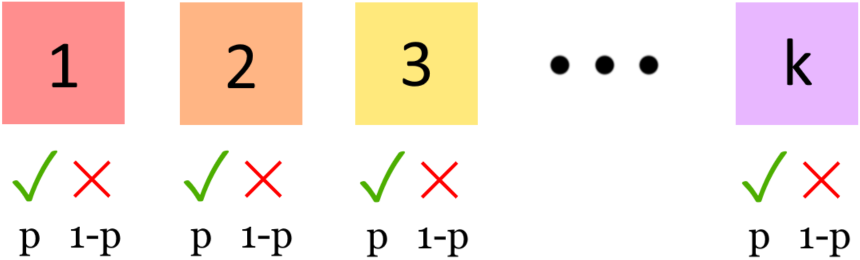
$\chi(G) \leq \text{degeneracy}(G) + 1$
$\chi_{\not\leftrightarrow}(G) \leq \text{degeneracy}(G) + 1$



When is $\chi_{\not\leftrightarrow}(G) \approx \sqrt{\text{degeneracy}(G)}$?

Random color inventory

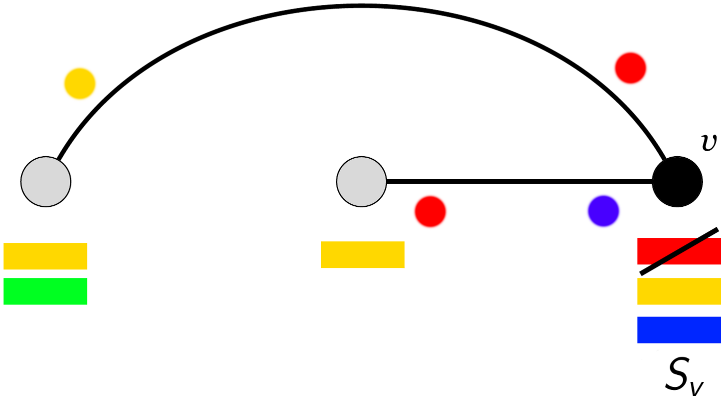
- Conflicts on $E(G)$ using $\{1, \ldots, k\}$ are given.
- We give each vertex a random *inventory* of colors.



Aharoni, Berger, Chudnovsky, Havet, Jiang 2018

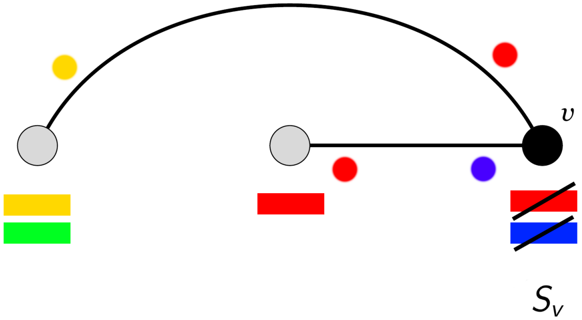
Random color inventory

If $c \in S_v$ is involved in a potential conflict with a color in a back-neighbor's inventory, then delete c from S_v.



Bad events

Bad event: all colors in S_v are deleted



- If no bad event occurs, then we can find a single-conflict coloring.

Lovász Local Lemma

- A set of bad events is given.
- Each bad event dependent with $< D$ other bad events.
- Each bad event has probability $\leq p$.

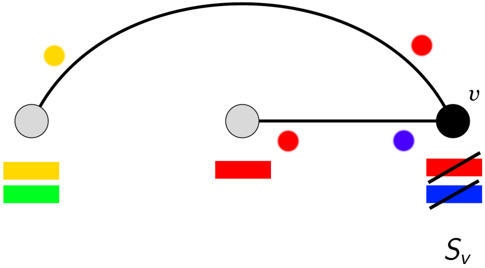
If

$$epD \leq 1,$$

then with positive probability, no bad event occurs.

$\chi_{\not\leftrightarrow}$ of degenerate graphs

- Consider G of max deg Δ and degeneracy d.



- $p < \exp\left(-\frac{k^2}{4d}\right)$, $D = (d+1)\Delta$
- $\chi_{\not\leftrightarrow}(G) = O(\sqrt{d \log(d\Delta)})$.

Summary of results

Question (Dvořák, Esperet, Kang, Ozeki 2018)

Is $\chi_{\not\leftrightarrow}(G) = O(\sqrt{d}\log n)$*?*

Theorem

If G is simple, then $\chi_{\not\leftrightarrow}(G) = O(\sqrt{d\log(d\Delta)})$.

Theorem

If G has edge-multiplicity μ, then $\chi_{\not\leftrightarrow}(G) = O(\sqrt{2^{\mu}\mu d\log(d\Delta)})$.

The answer is *yes*, if $\mu \leq \log\log n$.

Conclusion

Question (Dvořák, Esperet, Kang, Ozeki 2018)

Is $\chi_{\leftrightarrow}(G) = O(\sqrt{d} \log n)$ *when* μ *is large?*

Question

When else is $\chi_{\leftrightarrow}(G) \approx \sqrt{\chi(G)}$*?*

Thank you!