Parameterized complexity of fair deletion problems

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Deletion problems

Given a graph property P and graph G, vertex (edge) deletion problem is a task of finding set of vertices (edges) such that G after removing this set satisfies P.

Examples:

- Vertex Cover – $W \subseteq V$ such that $G \setminus W$ has no edge
- Feedback Vertex set – $W \subseteq V$ such that $G \setminus W$ is a forest
- Feedback Arc set – $F \subseteq E$ such that $G \setminus F$ is a DAG
- Odd cycle transversal – $W \subseteq V$ such that $G \setminus W$ is a bipartite
- Odd edge cycle transversal – $F \subseteq E$ such that $G \setminus F$ is a bipartite

For monotone properties finding any set is trivial.
We usually aim to find smallest such set.

Fair deletion problems

We usually aim to find smallest set S such that $G \setminus S$ satisfies P.

In Fair deletion problems, we want to find set that is locally small.

- For a set F of edges, we want to minimize $\deg(F)$
- For a set W of vertices, we want to minimize $|N(v) \cap W|$

Arises naturally in various context, e.g. defective coloring

Graph properties

We study properties definable in Monadic second order logic
In MSO, we can use

- logical connectives
- quantification over elements
- quantification over sets

(MSO_1)

Graph is described by set of vertices and adjacency relation $adj(\cdot, \cdot)$

(MSO_2)

Graph is described by set of vertices and edges and incidence relation $inc(\cdot, \cdot)$

Generalization of deletion problems

Sometimes we want to put additional restriction on the deleted set itself (for example CONNECTED VERTEX COVER)

One possibility: MSO formula with one free set variable, goal is to find set S such that $G \models \varphi(S)$ (in contrast to original $G \setminus S \models \psi$).

In this way, we can describe

- properties that S has to satisfy
- properties that $G \setminus S$ has to satisfy
- and more...

Still called deletion problem by some authors, though monadic second order evaluation is more appropriate.

Parameterized complexity

In parameterized complexity in addition to the input, we have a number called parameter.

Examples of parameters:

- size of the solution
- structural parameters (treewidth, cliquewidth, size of minimum vertex cover...)

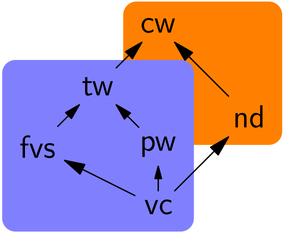
The running time is described as a function of both the size of the input and the parameter.

Complexity classes

- FPT (Fixed parameter tractable) – class of problems solvable in time $f(k)n^c$
- XP – class of problems solvable in time $n^{f(k)}$
- W[1]-hard

Overview of structural parameters

dense classes



sparse classes

Known results

- XP algorithm for generalized version of Fair MSO_2 edge deletion problem by Kolman, Lidický, and Sereni (running time $f(|\varphi|)n^{O(\mathrm{tw}(G))}$).
- Can be easily adapted to vertex version.

Hardness result for fair deletion

MSO FAIR DELETION is W[1]-hard with respect to treewidth.

Hardness result for fair deletion

MSO Fair deletion is W[1]-hard with respect to pathwidth + feedback vertex set size.

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Hardness result for fair deletion

FO FAIR DELETION is W[1]-hard with respect to pathwidth + feedback vertex set size.

Hardness based on ETH

It is known that MULTICOLORED CLIQUE cannot be solved in $f(k)n^{o(k)}$ unless Exponential Time Hypothesis fails (Chen et al. 2006).

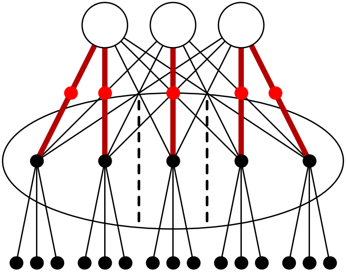
There is no algorithm for FAIR DELETION running in time $f(|\psi|, k)n^{o(\sqrt{k})}$ where k is the combined parameter $(\mathrm{tw}(G) + \mathrm{fvs}(G))$ unless Exponential Time Hypothesis fails

Proof idea (vertex version)

EQUITABLE PARTITION can be reduced to FAIR DELETION
(pick favorite problem: EQUITABLE COLORING, EQUITABLE CONNECTED PARTITION. . .)

- We want to delete vertices such that they encode a partition of vertices of G.
- The fair cost will be the size of the largest part.
- Minimal cost is achieved for equitable partition.

Proof idea

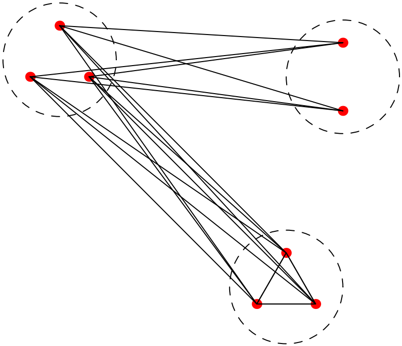


Neighborhood diversity

We want to partition $V(G)$ into sets $T_1, \dots, T_k$ such that

- each T_i is either a clique or an independent set, and
- between T_i, T_j is either a complete bipartite graph or no edge at all.

The smallest k such that the partition exists is the neighborhood diversity of G (denoted by $\mathrm{nd}(G)$).

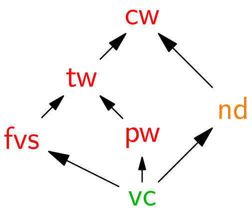


FPT algorithm with respect to nd

We allow only MSO_1 formulas (MSO_2 model checking is hard)

We build upon the result of Lampis used in MSO_1 model checking algorithm: Vaguely speaking the formula cannot distinguish between large cardinalities (counting $1, 2, 3, \mathrm{many}$)

Overview of the results



Green means FPT algorithm for both MSO_1 and MSO_2 deletion problems
Orange means FPT algorithm only for MSO_1
Red means hardness result

Open questions

- What is the complexity of deletion problems for parameterization by fair cost?
- Can the ETH based lower bound be improved to $f(|\psi|, k)n^{o(k)}$?
- Are there other structural parameters where can we obtain FPT algorithm?

Thank you!