Constant Congestion Brambles in Directed Graphs

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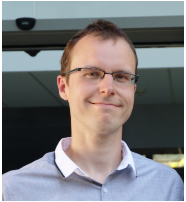


TU Berlin seminar 2021

Treewidth & Grid

Theorem (Undirected grid by Robertson & Seymour '86)

For every $k \geq 1$ there exists $t = f(k)$ such that every graph of treewidth at least t contains a $k \times k$ grid as a minor.

Theorem (Undirected grid by Chuzhoy & Tan '21)

For every $k \geq 1$ there exists $t = \mathcal{O}(k^9 \mathrm{polylog}(k))$ such that every graph of treewidth at least t contains a $k \times k$ grid as a minor.

Theorem (Directed grid Kawarabayashi & Kreutzer '15)

*For every $k \geq 1$ there exists $t = f(k)$ such that every **directed** graph of **directed** treewidth at least t contains a $k \times k$ **directed** grid as a minor.*

“Relaxed Grid” — Bramble

Definition

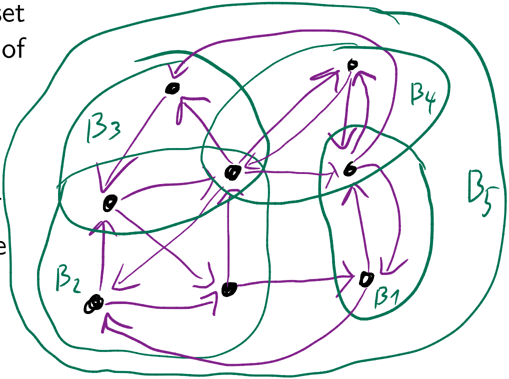
B_1, B_2, B_3, B_4, B_5

A **directed bramble** is a family of strongly connected subgraphs s.t.:

- every two subgraphs either **intersect in a vertex**, or
- the graph contains an **arc from one to the other** and an **arc back**.

- **Order**: **min** size of a vertex set that intersects every element of a bramble. 2
- **Size**: the number of its elements. 5
- **Congestion**: **max** number of elements that contain a single vertex. 4

For any bramble:
size $\leq$ order $\cdot$ congestion.



"Relaxed Grid" — Bramble

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Theorem (MPRzS '21)

For every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of ***congestion at most*** 8 *and size at least k.*

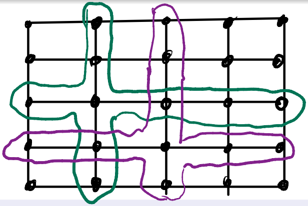
Brambles — Undirected Graphs

For any bramble: size $\leq$ order $\cdot$ congestion.

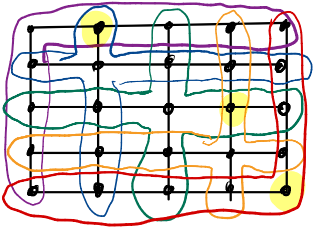
$k \times k$ grid contains a bramble of **order** k and **size** k^2, but **congestion** $\geq k-1$.

5×5 grid

$\forall_{i,j}$ row i
column j



$k \times k$ grid contains a bramble of **congetion** 2, **order** $\lceil k/2 \rceil$ and **size** k.



Order 3

History — Undirected Graphs

Theorem (Seymour, Thomas '93)

Max order of a bramble is ***exactly*** *its treewidth + 1.*

Theorem (Grohe, Marx '09 and Hatzel, Komosa, Pilipczuk, Sorge '20)

There are classes of graphs where for each $0 < \delta < 1/2$ any bramble of order $\tilde{\Omega}(k^{(0.5+\delta)})$ requires ***exponential*** *size in $k^{2\delta}$.*

For any bramble: size $\leq$ order $\cdot$ congestion.

History — Directed Graphs

Theorem (Reed '99)

Max order of a bramble is ***up to constant factor*** *its treewidth.*

Theorem (Grohe, Marx '09)

There are classes of graphs where for each $0 < \delta < 1/2$ any bramble of order $\tilde{\Omega}(k^{(0.5+\delta)})$ requires ***exponential*** *size in $k^{2\delta}$.*

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For every $k \geq 1$ there exists $t = \mathcal{O}(k^{48} \log^{13} k)$ such that every directed graph of directed treewidth at least t contains a bramble of ***congestion at most*** *8 and size at least k.*

History — Directed Graphs

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Directed grid give **congestion** 2 bramble of linear size.

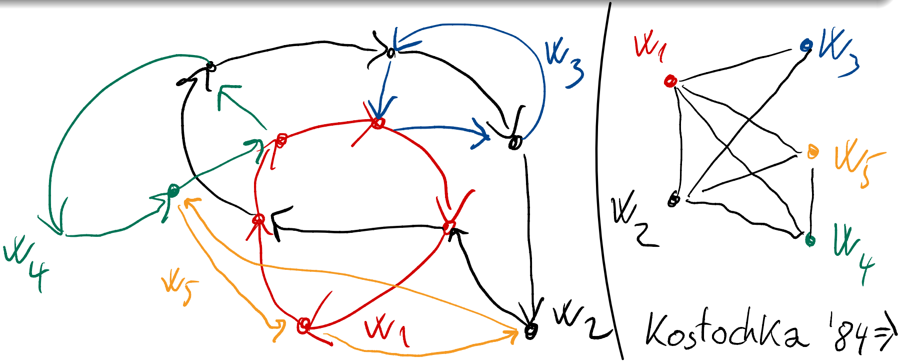
Hence,

- Kawarabayashi and Kreutzer '15 gives congestion 2 but **small-sized** bramble.
- Half-integral grid (Kawarabayashi, Kobayashi, Kreutzer '14) gives congestion 4 but **small-sized** bramble.
- **Planar graphs** grid (Hatzel, Kawarabayashi, Kreutzer '10) gives congestion 2 polynomial bound on a bramble.

Proof — Extracting a bramble (Dense case)

Lemma (Dense winning scenario)

Let c_{KT} be the constant from Kostochka '84. If a graph G contains a family $\mathcal{W}$ of ***closed walks*** *of congestion α, whose intersection graph is* ***not*** *$c_{KT} \cdot d \cdot \sqrt{\log d}$-degenerate, then G* ***contains a bramble of congestion*** *α* ***and size*** *d.*



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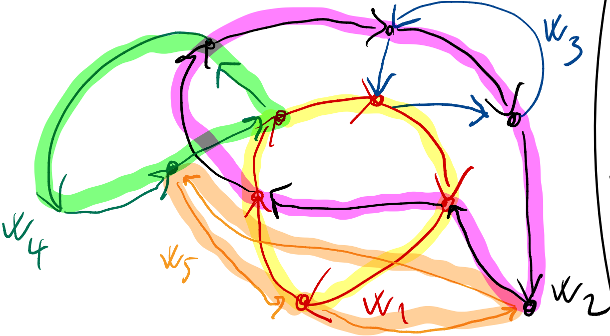
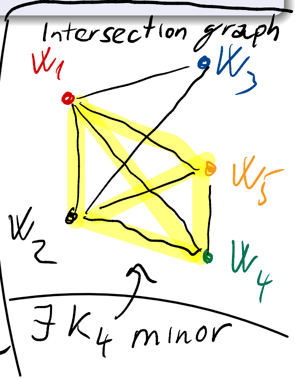
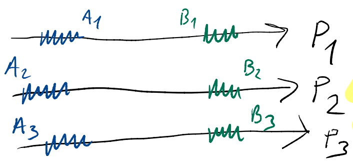


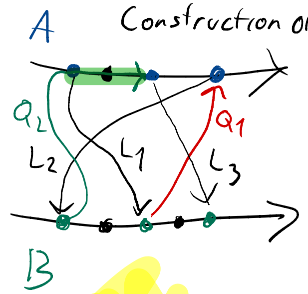


directed treewidth

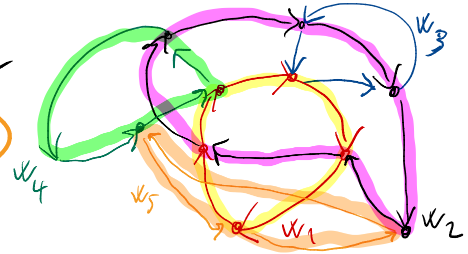
(a, b)-path system



Construction of threaded linkage



Untangled lemma



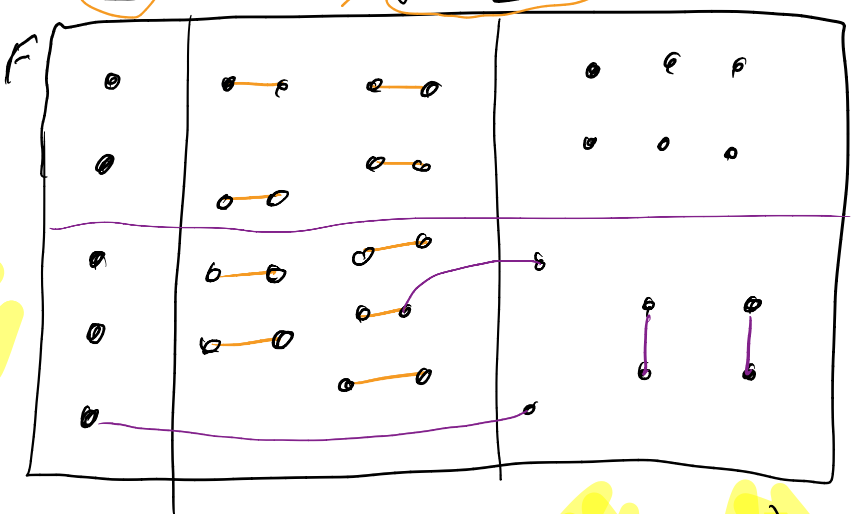
1

2 untangled → low congestion

Z

V − Z

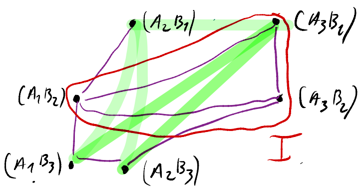
Main proof



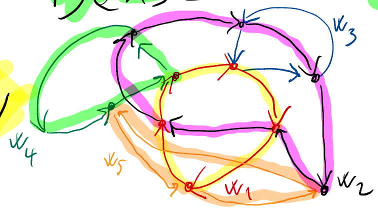
M_1

M_2

Bowtie Lemma

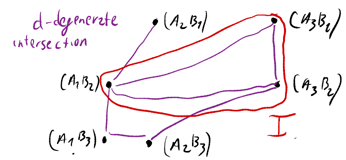


Dense winning



Bramble

Sparse winning scenarios



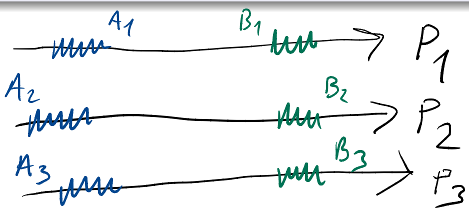
Proof — Starting Point

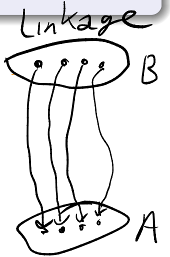
Definition (Path system)

Let $a, b \in \mathbb{N}$. An (a,b)-*path system* $(P_i, A_i, B_i)_{i=1}^{a}$ consists of

- vertex-disjoint paths $P_1, P_2, \ldots, P_a$, and
- for every $i \in [a]$, two sets $A_i, B_i \subseteq V(P_i)$, each of size b, such that every vertex of B_i appears on P_i later than all vertices of A_i,

such that $\bigcup_{i=1}^{a} A_i \cup B_i$ is **well-linked** in G.





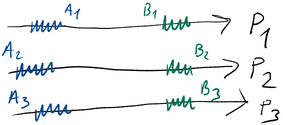
Kawarabayashi, Kreutzer '15:

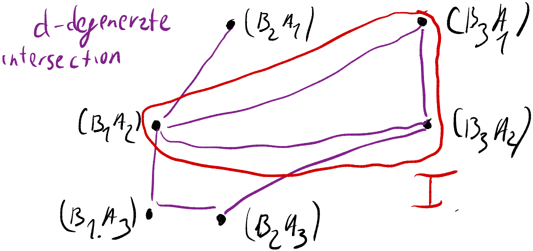
G **has directed treewidth** $ca^2b^2 \Rightarrow G$ **contains** (a,b)**-path system.**

Proof — Extracting a bramble (Sparse case)

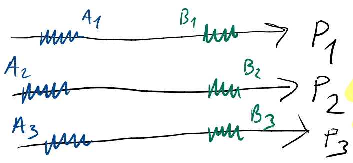
Lemma (Sparse winning scenario)

*$(P_i, A_i, B_i)_{i=1}^{a}$ be an (a,b)-path system, $\mathcal{I} \subseteq [a] \times [a] \setminus \{(i,i) \mid i \in [a]\}$, s.t. $|\mathcal{I}| \geq 0.6 \cdot a(a-1)$. The intersection graph of $\mathcal{L}_{i,j}$ and $\mathcal{L}_{i',j'}$ for every distinct $(i,j),(i',j') \in \mathcal{I}$ is **d-degenerate**. If $b > 4 \cdot e \cdot a^2 \cdot d$, **then G contains a bramble of congestion at most 4** and size $\geq c \cdot \left(\frac{a^{1/2}}{\log^{1/4} a}\right)$.*

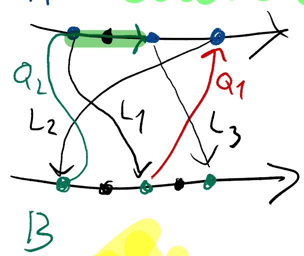




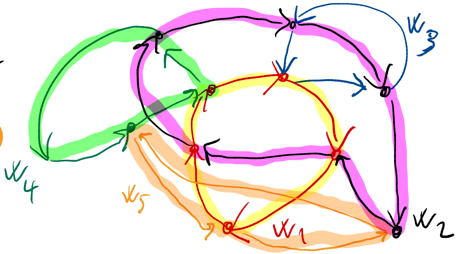
directed treewidth → (a,b)-path system



→ Construction of threaded linkage

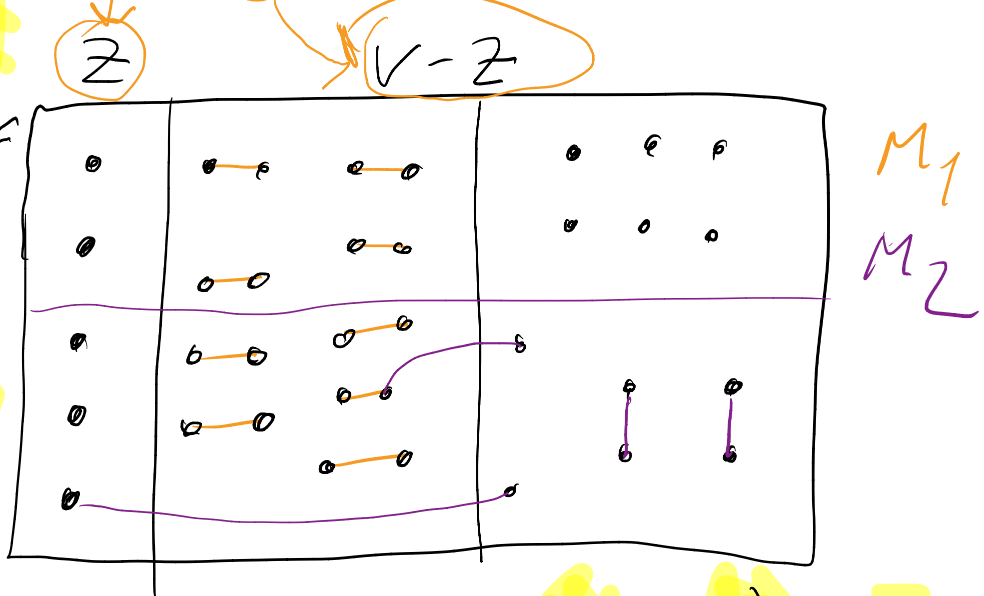


Untangled lemma

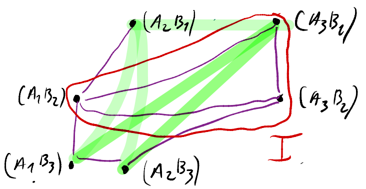


① ② untangled → low congestion

Main proof

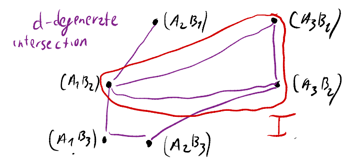


Bowtie Lemma



Dense winning

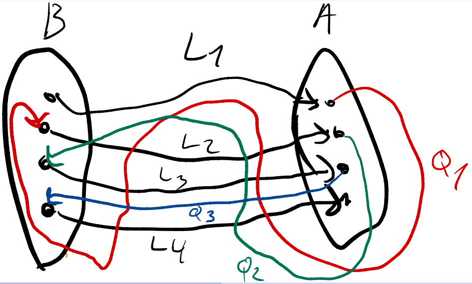
→ Bramble ← Sparse winning scenarios



Proof — Reduction of the congestion

Definition (Threaded linkage)

A **threaded linkage** is a pair $(W, \mathcal{L})$ where $\mathcal{L} = \{L_1, L_2, \ldots, L_\ell\}$ is a linkage and W is a walk such that there exist $\ell - 1$ paths $Q_1, Q_2, \ldots, Q_{\ell-1}$ such that W is the concatenation of $L_1, Q_1, L_2, Q_2, \ldots, Q_{\ell-1}, L_\ell$ in that order. The paths Q_i are called **threads**. A threaded linkage $(W, \mathcal{L})$ for $W = (L_1, Q_1, \ldots, Q_{\ell-1}, L_\ell)$ is **untangled** if for every i, the thread Q_i may only intersect the rest of W in L_i or L_{i+1}.



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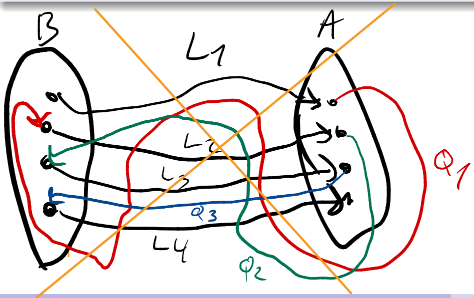
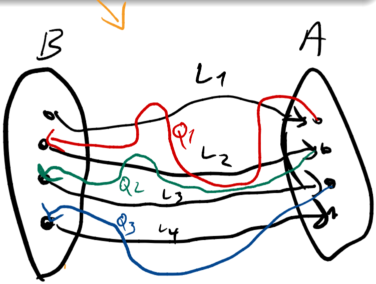




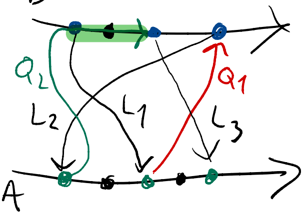
Proof — Reduction of the congestion

Lemma (Untangled threaded linkages)

Let $(W, \mathcal{L})$ be a threaded linkage of size b and of overlap α. Let $x, d \in \mathbb{N}$ such that $b \geq xd + (d-1)$. Then one of the following exists:

1. *A family $\mathcal{Z}$ of d closed walks, such that for every walk $W \in \mathcal{Z}$ there exists a distinct path $P(W) \in \mathcal{L}$ that is a subwalk of W, and $\mathcal{Z}$ has overlap α; or*
2. *an untangled threaded linkage $(W', \mathcal{L}')$ where W' is a subwalk W and $\mathcal{L}' \subseteq \mathcal{L}$ is of size at least x. In particular, $(W', \mathcal{L}')$ is of overlap α.*

Construction of threaded linkage



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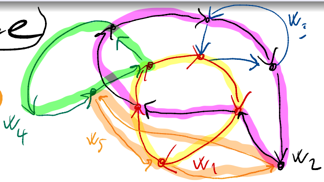
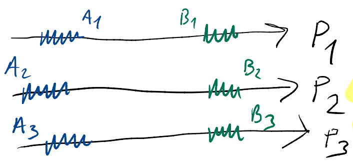


2 untangled → low congestion

directed treewidth

(a, b)-path system



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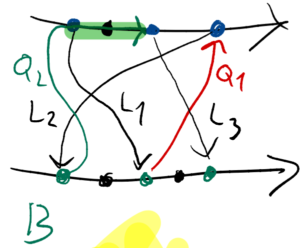
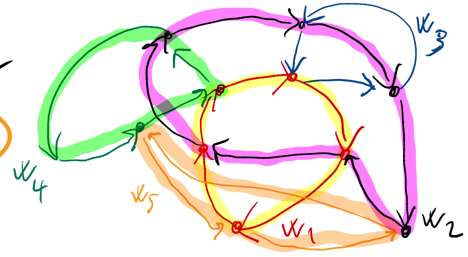


Untangled lemma



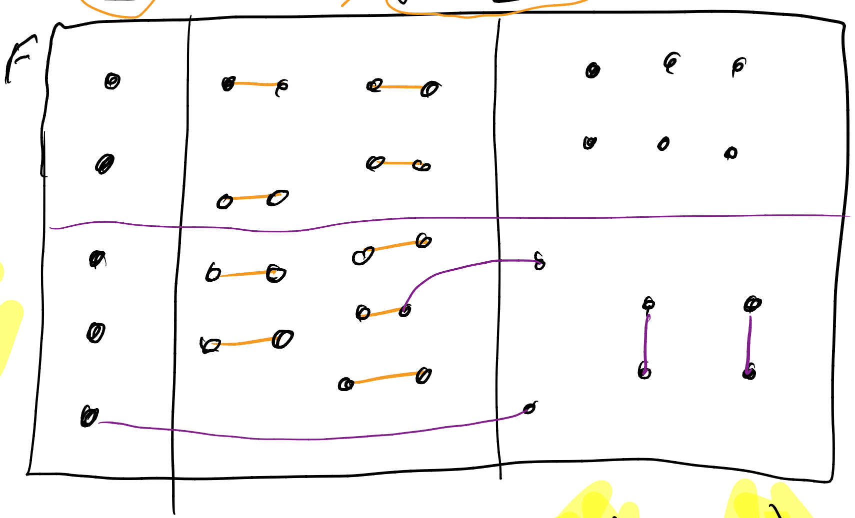
1

2 untangled ~> low congestion

Main proof

Z

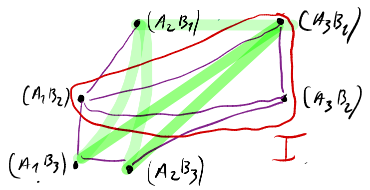
V − Z



M_1

M_2

Bowtie Lemma

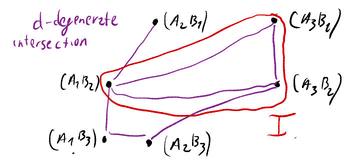


Dense winning

Bramble

Sparse winning scenarios

d-degenerate intersection

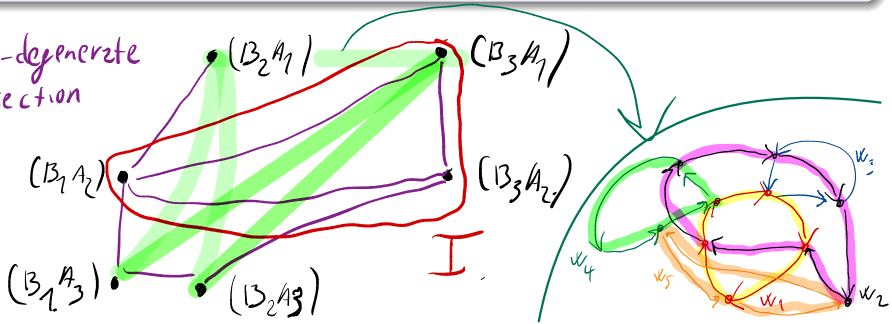


Proof — Bowties

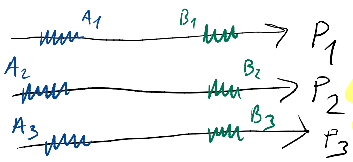
Lemma (Bowtie lemma)

Let $(W_1, \mathcal{L}_1)$ and $(W_2, \mathcal{L}_2)$ be two threaded linkages of overlap α and β, such that the intersection graph $I(\mathcal{L}_1, \mathcal{L}_2)$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ is ***not $(2^9 \cdot 5 \cdot d)$-degenerate****. Then there is a family $\mathcal{Z}$ of d* ***closed walks*** *such that every walk in $\mathcal{Z}$* ***contains at least one path of $\mathcal{L}_1$ and one path of $\mathcal{L}_2$*** *as a subwalk, and the congestion of $\mathcal{Z}$ is at most $\alpha + \beta$.*

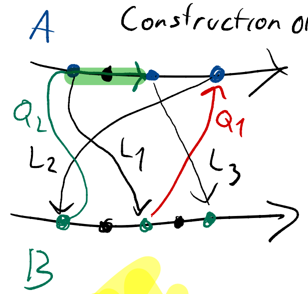
d-degenerate
intersection



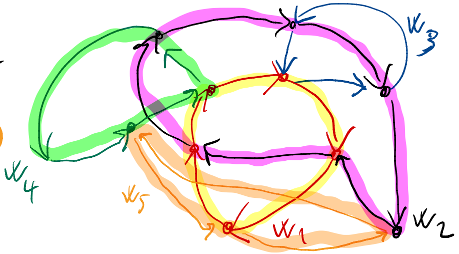
directed treewidth → (a,b)-path system



→ Construction of threaded linkage



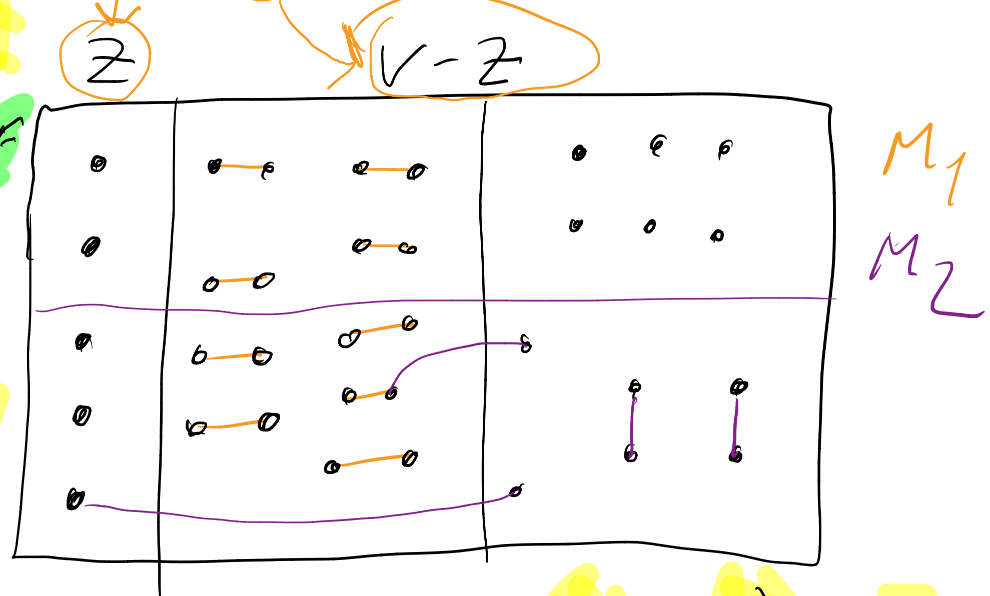
Untangled lemma



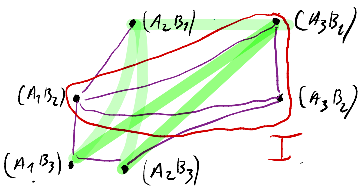
1

2 untangled → low congestion

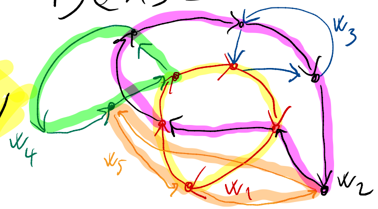
Main proof



Bowtie Lemma

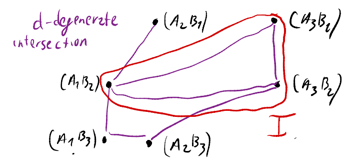


Dense winning



→ Bramble ←

Sparse winning scenarios

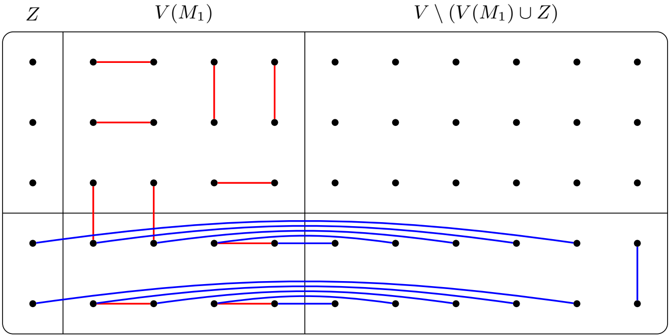


Main Proof — Setup

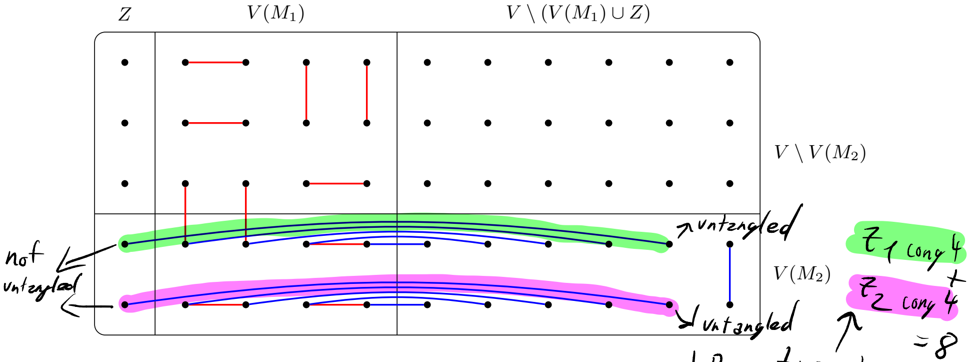
Each vertex represent (B_i, A_j) linkage. $Z \subseteq V$ be linkages s.t. untangled lemma results in (1) outcome: a **family of closed walks $\mathcal{Z}$ of overlap 3**.
M_1 **be a maximum matching** in $H_1 - Z$, where edges representes linkages with intersection graph that is **not d_1-degenerate**.
M_2 **be a maximum matching** in graph $(V, E(H_2) \setminus \binom{V(M_1) \cup Z}{2})$, where edges representes linkages with intersection graph that is **not d_2-degenerate**

$d_2 \leq d_1$

$H_1 \subseteq H_2$



Main Proof (Dense Case)



+Bowtie Lemma

Lemma (Dense winning scenario)

Let c_{KT} be the constant from Kostochka '84. If a graph G contains a family $\mathcal{W}$ of ***closed walks*** *of congestion α, whose intersection graph is* ***not*** *$c_{KT} \cdot d \cdot \sqrt{\log d}$-degenerate, then G* ***contains a bramble of congestion*** α ***and size*** d.

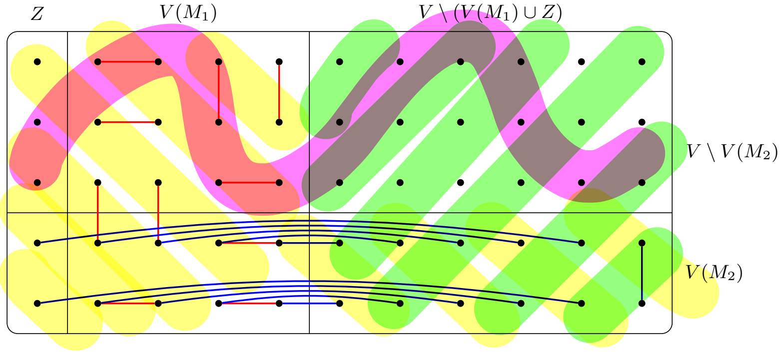
Main Proof (Three Sparse Cases)

At least one of the following cases occurs:

Case 1. $|V \setminus (V(M_1) \cup Z)| \geq 0.6|V|$;

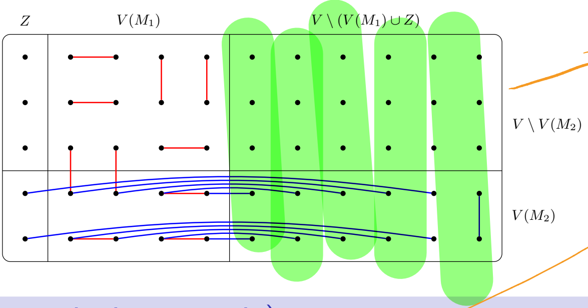
Case 2. $|V(M_1) \cup V(M_2) \cup Z| \geq 0.6|V|$;

Case 3. $|V \setminus V(M_2)| \geq 0.6|V|$.



Main Proof (Three Sparse Cases) — Case 1.

$$|V \setminus (V(M_1) \cup Z)| \geq 0.6|V|$$



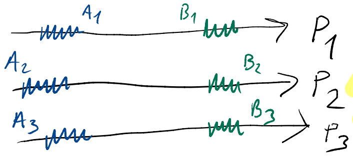
Indep. set in H_1

Lemma (Sparse winning scenario)

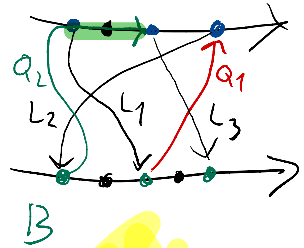
*$(P_i, A_i, B_i)_{i=1}^{a}$ be an (a, b)-path system, $\mathcal{I} \subsetneq [a] \times [a] \setminus \{(i, i) \mid i \in [a]\}$, s.t. $|\mathcal{I}| \geq 0.6 \cdot a(a-1)$. The intersection graph of $\mathcal{L}_{i,j}$ and $\mathcal{L}_{i',j'}$ for every distinct $(i, j), (i', j') \in \mathcal{I}$ is **d-degenerate**. If $b > 4 \cdot e \cdot a^2 \cdot d$, **then G contains a bramble of congestion at most 4** and size $\geq c \cdot \left(\frac{a^{1/2}}{\log^{1/4} a}\right)$.*

directed treewidth

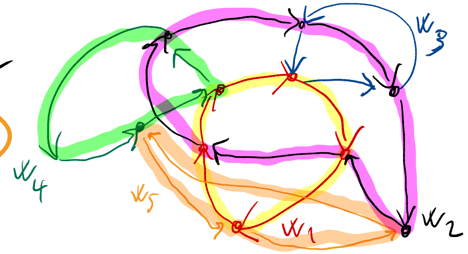
(a, b)-path system



Construction of threaded linkage



Untangled lemma



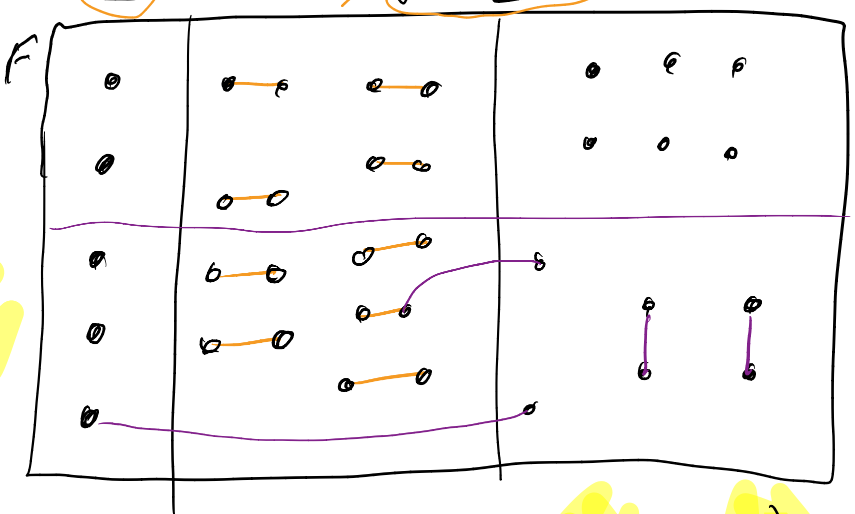
1

2 untangled → low congestion

Z

V − Z

Main proof

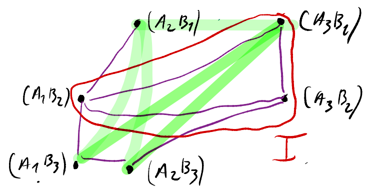


M_1

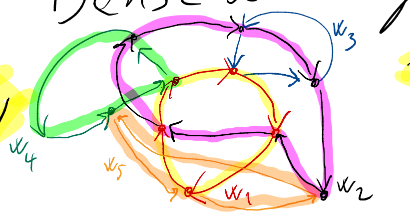
M_2

Thank You

Bowtie Lemma



Dense winning



Bramble

Sparse winning scenarios