Simplified Algorithmic Metatheorems Beyond MSO

Treewidth and Neighborhood Diversity

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Monadic Second Order Logic – MSO_1

- universe is a graph
- variables for elements (vertices, edges) x,y
- variables for sets of vertices X,Y

$$\begin{aligned}(\exists R, G, B)\,(R \cup G \cup B = V) &\wedge (\forall u, v \in R \rightarrow \{u, v\} \notin E)\\ &\wedge (\forall u, v \in G \rightarrow \{u, v\} \notin E)\\ &\wedge (\forall u, v \in B \rightarrow \{u, v\} \notin E)\end{aligned}$$

MSO_2

- adds a possibility for variables for sets of **edges**
- model checking is hard on graphs of bounded neighborhood diversity

Monadic Second Order Logic

Theorem (Courcelle)

There exists an algorithm that, given

- *an* MSO *formula* φ, *and*
- *an n-vertex graph* G *with its tree decomposition of width* t *and evaluation of all the free variables of* φ,

verifies whether φ *is satisfied in* G *in time* $f(|\varphi|, t) \cdot n$.

- φ in prenex form
- $|\varphi|$ is the number of quantifiers in φ

Extending MSO – Local Cardinality Constraints

- linear optimization FPT(tw) [Arnborg, Lagergren and Seese 91]
- further objectives FPT(tw) [Courcelle and Mosbah 93]
- fair objective function XP(tw) [Kolman, Lidický and Sereni 09]
  - given MSO formula $\varphi(X)$ with a free vertex set variable X
  - minimizes maximum degree in the subgraph given by X

$$\min_{X \subseteq V} \max_{v \in V} \{X(v) \mid \varphi(X)\}$$

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  - W[1]-hard(tw) FPT(nd) [Masařík, Toufar 16]

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- add a "global" relation $[|X| \geq |Y|^2]$
- cardinality constraints FPT (nd) [Ganian, Obdržálek 11]

Equitable k-Coloring

Input: graph $G = (V, E)$

Task: find a partition $V_1, \ldots, V_k$ of V such that $G[V_i]$ is an edgeless graph and $\big||V_i| - |V_j|\big| \leq 1$ or report that no such partition exists

$\mathrm{MSO}^{\mathrm{G}}_{\mathrm{lin}}$ only **linear relations** added $|X| \leq |Y| + 7; |Y| = 2|X|$

$\mathrm{MSO}^{\mathrm{G}}$ general relations added $|X|^2 + |Y|^3 = |Z|$

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Indeed, we can combine all possible extensions!

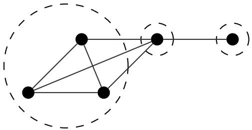
Neighborhood Diversity

We say that two (distinct) vertices u, v are of the same *neighborhood type* if they share their respective neighborhoods, that is when

$$N(u) \setminus \{v\} = N(v) \setminus \{u\}.$$

Definition (Lampis 12)

A graph $G = (V, E)$ has *neighborhood diversity* at most w ($\mathrm{nd}(G) \leq w$) if there exists a partition of V into at most w sets such that all the vertices in each set have the same neighborhood type.



Extending MSO – Our Results

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Vertex Cover – Hardness Reduction

k-Multicolored Clique *Parameter: k*

Input: k-partite graph $G = (V_1 \dot\cup \cdots \dot\cup V_k, E)$, where V_a is an independent set for every $a \in [k]$.

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Input: Graph $G = (V, E)$ with $|V| = n$ and a function $f\colon V \to 2^{[n]}$.

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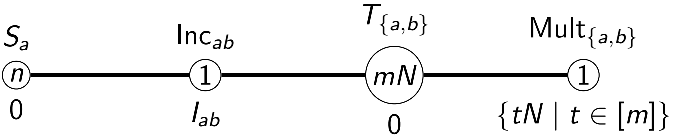

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Neighborhood Diversity – $\mathrm{MSO}^{\mathrm{GL}}_{\mathrm{lin}}$

- let φ be a $\mathrm{MSO}^{\mathrm{G}}_{\mathrm{lin}}$ formula with free set variables $X_1, \ldots, X_\ell$
- φ contains q_S set quantifiers and q_e element quantifiers
- let $G = (V, E)$ be a graph with $\mathrm{nd}(G) = \nu$ with types $T_1, \ldots, T_\nu$

Definition

Let $\mu\colon \{X_1, \ldots, X_\ell\} \to 2^V$ be a variable assignment. The signature of μ is a mapping from $S_\mu\colon [\nu] \times 2^{[\ell]} \to \mathbb{N}$ defined by $S_\mu(j, I) = \left|\bigcap_{i \in I} \mu(X_i) \cap T_j\right|$.

Lemma (Lampis 12)

Suppose that μ, μ' are two variable assignments such that for every $I \subseteq [\ell]$, $j \in [\nu]$ we have either

- $S_\mu(j, I) = S_{\mu'}(j, I)$, *or*
- *both* $S_\mu(j, I), S_{\mu'}(j, I) > 2^{q_S} \cdot q_e$.

It follows that $G, \mu \models \varphi$ if and only if $G, \mu' \models \varphi$.

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Theorem (Lenstra 83, Frank&Tardos 87)

Let p be the number of integral variables in a (mixed) integer linear program and let L be the number of bits needed to encode the program. Then it is possible to find an optimal solution in time $O\big(p^{2.5p}\,\mathrm{poly}(L)\big)$.

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2. We view this LP as an ILP and construct an equivalent constraint satisfaction problem instance J' of bounded treewidth.
3. Our CSP have two types of constraints hard $(\mathcal{H}')$ and soft $(\mathcal{S})$.
4. We show that if $\mathcal{H}'$ and $\mathcal{S}$ have the **local scope property** (the scope of all constraints is restricted to variables corresponding to the descendants of some node of the treedecomposition), it is possible to add new constraints derived from $\mathcal{H}'$ and $\mathcal{S}$ to instance J' which results in instance J, such that J is an extension of I.

Conclusions

Open problems:

- what further extensions of MSO admit an FPT algorithm on graphs with bounded neighborhood diversity
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