On 3-Coloring of $(2P_4, C_5)$-Free Graphs

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Hereditary Graph Classes

A graph class is *hereditary* if it is closed under vertex deletion.

- uniquely characterized by a minimal (not necessarily finite) set of forbidden induced subgraphs

A graph is:

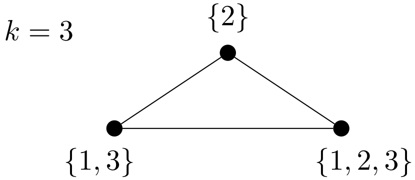
- H*-free* if it does not contain H as an **induced** subgraph.
- $(H_1, \ldots, H_k)$*-free* if it is H_1-free **and, . . . , and** H_k-free.

Coloring

- A (proper) **coloring** of a graph is an assignment of colors to vertices such that no two adjacent vertices get the same color.

k-Coloring:
Given: Graph G,
Task: Does there exists a coloring of G which uses at most k colors?

List k-Coloring:
Given: Graph G and a palette $P(v) \subseteq \{1, \ldots, k\}$ for any vertex v,
Task: Does there exist a coloring of G that respects P?



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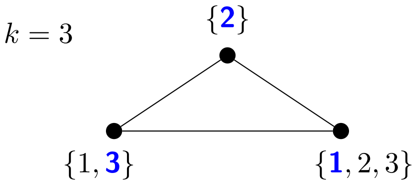
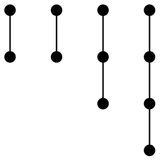


State-of-the-art k-Coloring on H-Free Graphs

- For every $k \geq 3$, k-Coloring on H-free graphs is **NP-complete** if
  - H contains a **cycle**, or **[Emden-Weinert, Hougardy, Kreuter 98]**
  - an induced **claw**. **[Holyer 81 & Leven, Galil 83]**
- Hence, it remains to consider the case where H is a **disjoint union of paths.**

  Example (disjoint union of paths):



$H = 2P_2 + P_3 + P_4$

State-of-the-art P_t-free graphs

	k-Coloring			
t	$k=3$	$k=4$	$k=5$	$k \geq 6$
$t \leq 5$	P	P	P	P
$t = 6$	P	P	NP-c	NP-c
$t = 7$	P	NP-c	NP-c	NP-c
$t \geq 8$	?	NP-c	NP-c	NP-c

	List k-Coloring			
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$t = 7$	P	NP-c	NP-c	NP-c
$t \geq 8$	?	NP-c	NP-c	NP-c

3-coloring on P_t-free graphs is quasi-polynomial
[Pilipczuk, Pilipczuk, Rzążewski '20]

- '08 Hoang, Kaminski, Lozin, Sawada, Shu
- '16 Huang
- '04 Randerath, Schiermeyer
- '12 Broersma, Fomin, Golovach, Paulusma
- '19 Spirkl, Chudnovsky, Zhong
- '14 Golovach, Paulusma, Song
- '17 Bonomo, Chudnovsky, Maceli, Schaudt, Stein, Zhong

3-Coloring on H-free graphs For H Linear Forest

List 3-Coloring is polynomial on H-free graphs for $|V(H)| \leq 7$

- sP_3-free graphs **[Broersma, Golovach, Paulusma, Song '12]**
- 3-Coloring on H-free graphs polynomial $\rightarrow$ 3-Coloring on $H + P_1$-free graphs polynomial **[Broersma, Golovach, Paulusma, Song '12]**
- $rP_1 + P_5$-free graphs, $r \geq 0$ **[Couturier, Golovach, Kratsch, Paulusma '15]**
- P_7-free graphs **[Bonomo, Chudnovsky, Maceli, Schaudt, Stein, Zhong '17]**
- $P_3 + P_4$-free and $P_2 + P_5$-free graphs. **[Klimošová, Malík, TM, Novotná, Paulusma, Slívová '20]**

List 3-Coloring is polynomial on $rP_3 + P_6$ **[Chudnovsky, Huang, Spirkl, Zhong '20]**

List 3-Coloring is polynomial on H-free graphs for $|V(H)| \leq 8$ if

- $H \neq P_8$
- $H \neq 2P_4$

More Forbidden Graphs

3-Coloring is polynomial on:

- (P_t, C_4)-free graphs **[Golovach, Paulusma, Song '14]**
- (P_8, C_3, C_4)-, (P_8, C_3, C_5)-, (P_8, C_4, C_5)-free graphs **[Chudnovsky, Stacho '18]**
- (P_9, C_3, C_5)-free graphs **[Rojas, Stein '20]**
- $s, t \geq 1$; $(SDK_{1,s}, P_t)$-free graphs **[Chudnovsky, Sprikl, Zhong '20]**

Theorem (JKMNP '21)

*The 3-Coloring problem is polynomial-time solvable on $(2P_4, C_5)$-**free** graphs.*

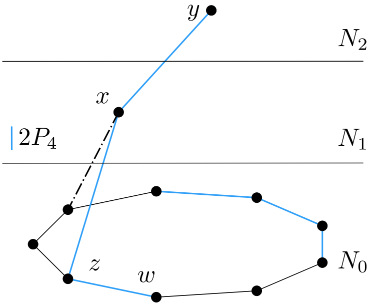
Proof High Level Sketch

- Constant size subgraph as N_0; **try all possible colorings of N_0**.
- Partition the rest of the graph into two sets N_1 (**neighbours of N_0**) and N_2 (the rest).
- Once N_0 contains an induced P_4 **then** N_2 has to be **P_4-free**.
  List 3-coloring is **polynomial-time** solvable on P_4-free graphs.
- Reductions & branching to **polynomially** many instances.
- Solve each final instance by **2-SAT** (2-coloring) algorithm.
- The initial solution is a yes-instance **iff** at least one branch is a yes-instance.

Suppose that G is $(2P_4, C_5, C_7)$-free.

- K_4 and $\overline{C_7}$ are not 3-colorable.
- As K_4-free, hence also $\overline{C_{\geq 8}}$-free.
- As $2P_4$-free, hence also $C_{\geq 10}$-free.
- If also C_9-free then $(C_{odd \geq 5}, \overline{C_{odd \geq 5}})$**-free** that is **perfect**.
- Hence, Polynomial-time solvable **[Grötschel, Lovász, Schrijver '84]**

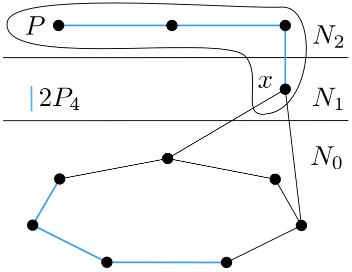
Therefore, there is an induced C_9.

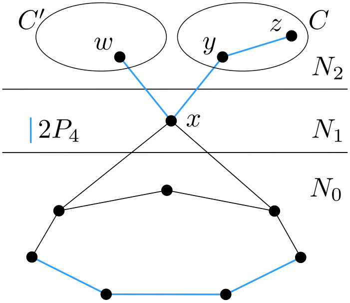


C_7 as N_0—The Initial Situation

After fixing the coloring of N_0 and applying all available basic reductions, the graph G has the following properties.

- Each vertex x of N_1 satisfies either $N_0(x) = \{v_i\}$ for some i, or $N_0(x) = \{v_i, v_{i+2}\}$ for some i.
- **Each induced copy of P_4 in G has at most two vertices in N_2.**
- G is connected.



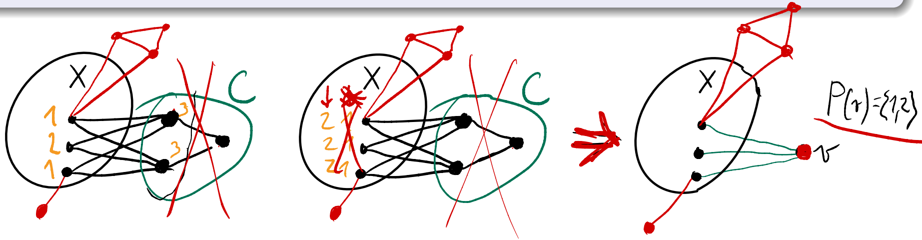


Involved Reductions—Cut Reduction

Cut Reduction

Let $X \subseteq V$ be a vertex cut of G, let C be a union of one or more connected components of $G - X$. Moreover, it holds:

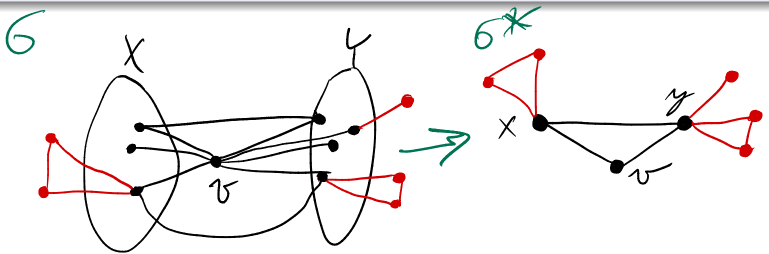
- C has at least two vertices.
- X is an **independent** set in G.
- $\forall x \in X$ have the same palette $P(x) = \{1, 2\}$.
- For any two vertices x, x' in X, we have $N(x) \cap C = N(x') \cap C$.
- The graph $C \cup X$ is P_4-**free**.



Involved Reductions-Neighborhood Collapse

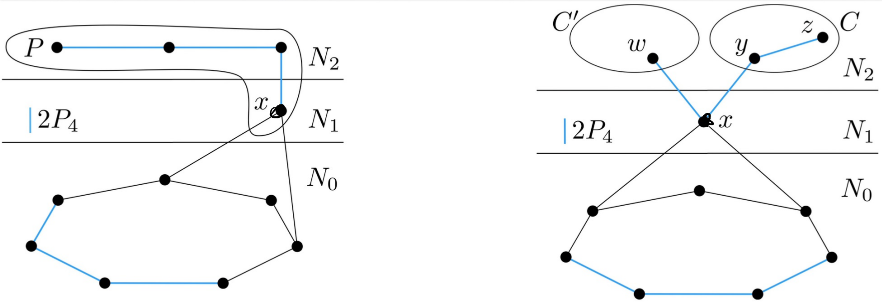
Neighborhood Collapse

Let $v \in G$ and $N(v)$ is a connceted bipartite graph (X, Y) with vertices in X (or Y) of the same palette. **Then $G*$ has X replaced by x and Y by a vertex y.**



Neighbourhood collapse preserves $2P_4$-freeness.

Proof High Level Sketch



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Conclusions & Open problems — 3-coloring

3-coloring

- $(2P_4, C_3)$-free graphs
- (P_8, C_3)-free graphs
- (P_8, C_5)-free graphs

5-coloring

The 5-coloring problem is NP-complete on $(2P_4)$-free graphs

[Hajebi, Li, Spirkl '21]

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- (P_8, C_5)-free graphs

5-coloring

The 5-coloring problem is NP-complete on $(2P_4)$-free graphs

[Hajebi, Li, Spirkl '21]

Thank you!