Proving a directed analogue of the Gyárfás-Sumner conjecture for orientations of P_4

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Dagstuhl seminar 22481
Dagstuhl 2022

Gyárfás-Sumner conjecture

No induced subgraph H in G

χ-bounding

We say that a graph H is χ-bounding if $\exists$ a function f s.t. $\forall$ H-**free** graph G:

$$\chi(G) \leq f(\omega(G)).$$

Conjecture (The Gyárfás-Sumner conjecture 1975$\sim$81)

Every forest is χ-bounding.

Erdős 1959*:* Every χ-bounding graph is a forests.

Scott and Seymour 2020: A survey of χ-boundedness.

What about digraphs?

χ-bounding

We say that a **di**graph H is χ-bounding if $\exists$ a function f s.t. $\forall$ H-free **di**graph G:

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Every oriented forest is χ-bounding.

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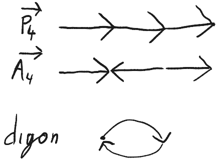
maxclique size of underlying graph

Conjecture (Directed Gyárfás-Sumner?)

Every oriented forest is χ-bounding.

NO if $H = \overrightarrow{P_4}$ **Kierstead & Trotter 1991**, or
NO if $H = \overrightarrow{A_4}$ **Gyárfás 1990**

NO if H contains a **digon**
NO if G contains a **digon** (while H nontrivial)



What about digraphs?

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We say that an **oriented (or)**graph H is χ-bounding if $\exists$ a function f s.t. $\forall$ H-free **or**graph G:

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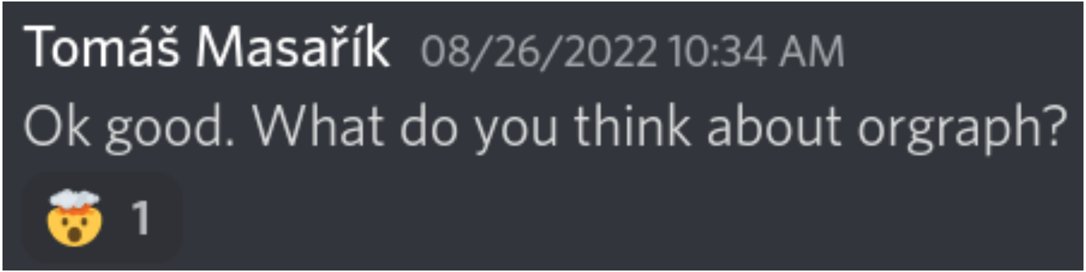

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What about orgraphs? Let's use dichromatic number.

Definition (dichromatic number)

The dichromatic number of D, denoted as $\vec{\chi}(D)$, is the minimum number of colors needed for a dicoloring of D.

Partition of vertices s.t. there is NO directed cycles in any part (color).

$\vec{\chi}$-bounding

We say that an **or**graph H is $\vec{\chi}$-bounding if $\exists$ a function f s.t. $\forall$ H-free **or**graph G:

$$\vec{\chi}(G) \leq f(\omega(G)).$$

Conjecture (Aboulker, Charbit & Naserasr 2021)

Every orientation of a forest is $\vec{\chi}$-bounding.

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Conjecture (Aboulker, Charbit & Naserasr 2021)

Every orientation of a forest is $\overrightarrow{\chi}$-bounding.

Harutyunyan and Mohar 2012: Every $\overrightarrow{\chi}$-bounding graph is a forests.
In particular, **NO digons** are allowed.

Chudnovsky, Scott & Seymour 2019 showed that $\rightarrow\leftarrow\leftarrow$, $\leftarrow\leftarrow\rightarrow$, and oriented stars are even χ-**bounding**. Let T be any fixed orientation of K_3. **Aboulker, Charbit & Naserasr 2021**: $(T, \overrightarrow{P_4})$-free has **bounded** $\overrightarrow{\chi}$.

Let K_t denote the transitive tournament on t vertices. **Steiner 2021**: $(K_3, \overrightarrow{A_4})$-free oriented graphs has **bounded** $\overrightarrow{\chi}$.

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Aboulker, Charbit & Naserasr 2021: $(T, \vec{P_4})$-free has **bounded** $\vec{\chi}$.

Steiner 2021: $(K_3, \vec{A_4})$-free oriented graphs has **bounded** $\vec{\chi}$.

Theorem (CMPRS 2022+)

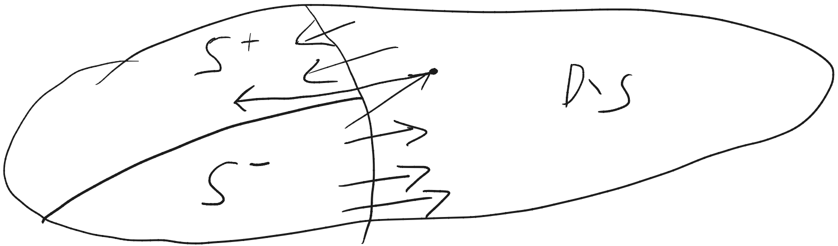
Let H be an oriented P_4. Then, the class of H-free orgraphs is $\vec{\chi}$-bounded.
In particular, for any H-free orgraph D,

$$\vec{\chi}(D) \leq (\omega(D) + 7)^{(\omega(D)+8.5)}.$$

Proof sketch—Dipolar set

Definition (dipolar set Aboulker, Charbit & Naserasr 2021)

A dipolar set of an orgraph D is a nonempty subset $S \subseteq V(D)$ that can be partitioned into S^+, S^- such that no vertex in S^+ has an **out-neighbor in** $V(D \setminus S)$ and no vertex in S^- has an **in-neighbor** in $V(D \setminus S)$.



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Lemma (Aboulker, Charbit & Naserasr 2021)

*Let $\mathcal{D}$ be a family of **orgraphs closed under taking induced subgraphs**. Suppose there exists a constant c such that every $D \in \mathcal{D}$ has a dipolar set S with $\overrightarrow{\chi}(S) \leq c$. Then every $D \in \mathcal{D}$ satisfies $\overrightarrow{\chi}(D) \leq 2c$.*

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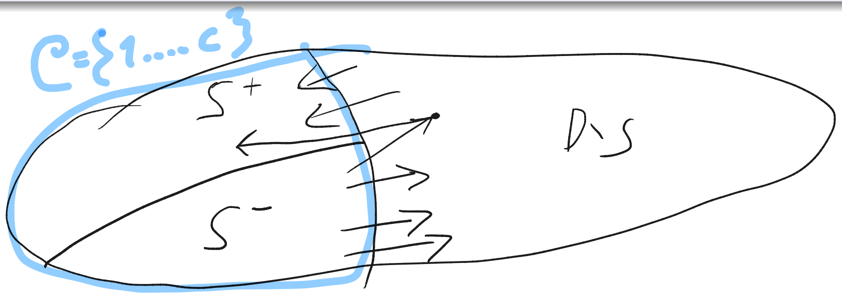

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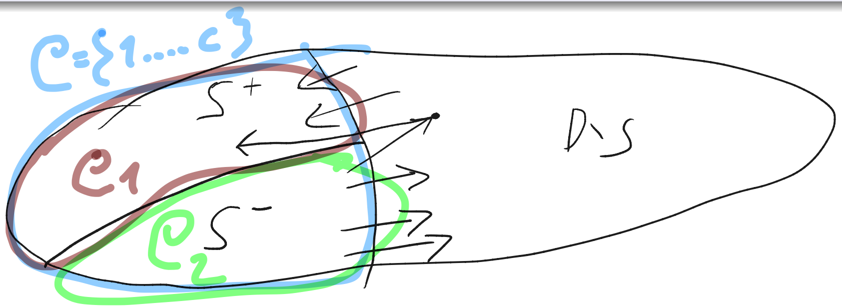

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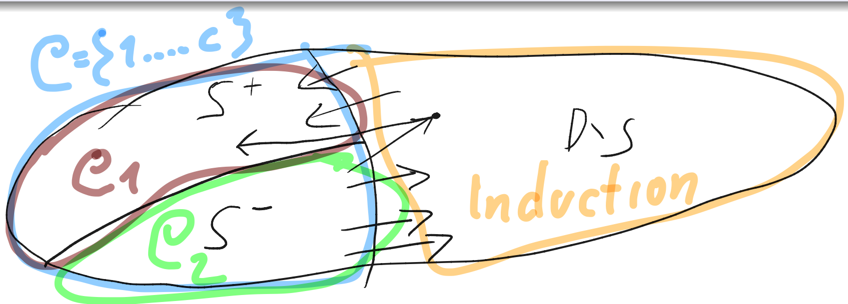

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Let $\mathcal{D}$ be a family of ***orgraphs closed under taking induced subgraphs****. Suppose there exists a constant c such that every $D \in \mathcal{D}$ has a dipolar set S with $\overrightarrow{\chi}(S) \leq c$. Then every $D \in \mathcal{D}$ satisfies $\overrightarrow{\chi}(D) \leq 2c$.*

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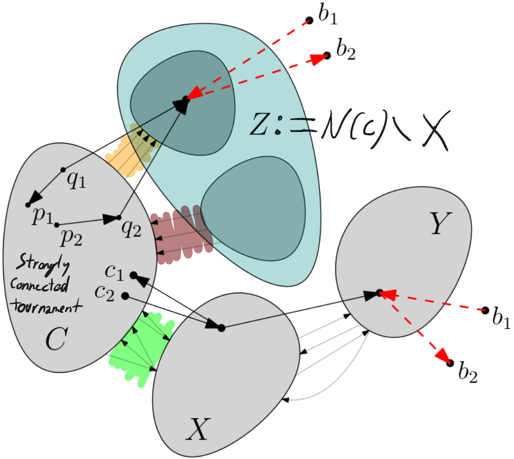
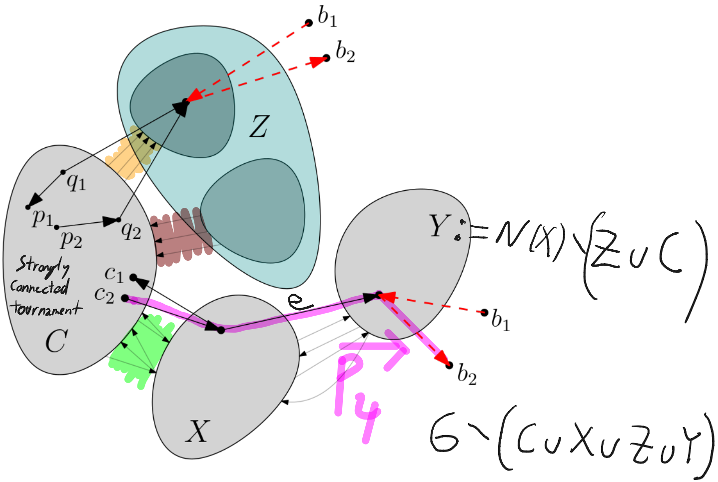


Proof sketch—Finding a dipolar set and the strongly connected case

If G contains strongly connected tournament of size $\omega(G)$:



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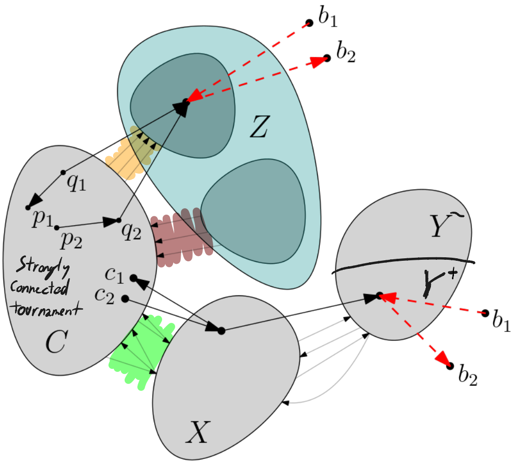
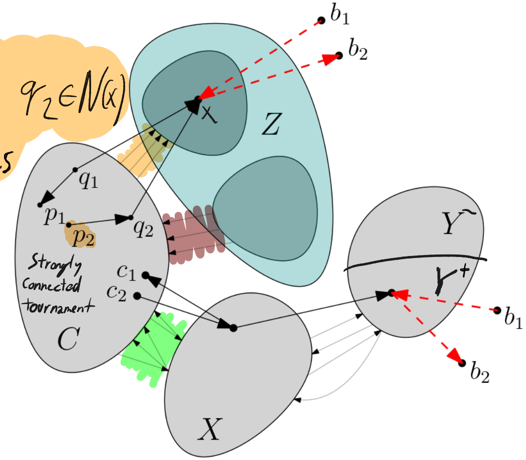


$G \setminus (C \cup X \cup Z \cup Y)$

If G contains strongly connected tournament of size $\omega(G)$:

$\exists p_2, q_2$ s.t. $p_2 \notin N(x) \wedge q_2 \in N(x)$
$(p_2, q_2) \in$ Arcs



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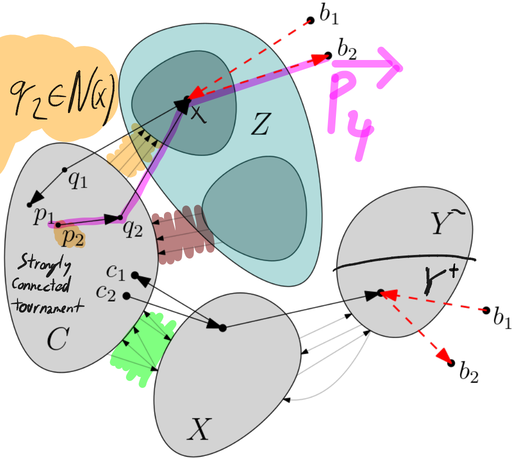

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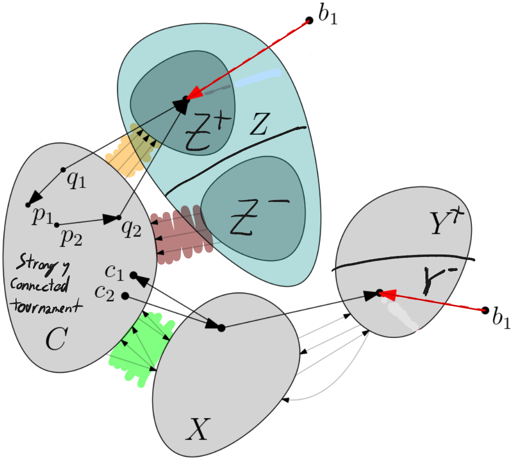
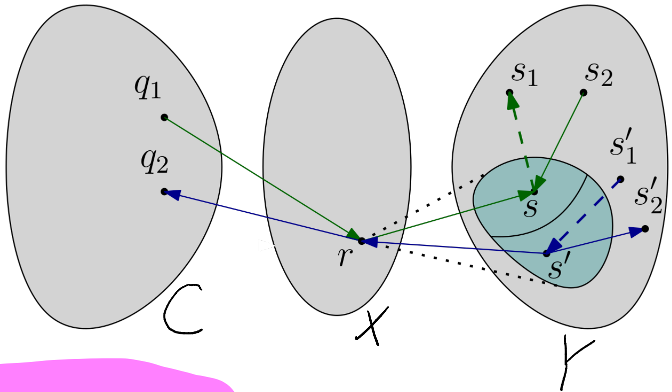


$G \setminus (C \cup X \cup Z \cup Y)$

$\Rightarrow$ $C \cup X \cup Z \cup Y$ is a dipolar set!

If G contains strongly connected tournament of size $\omega(G)$:



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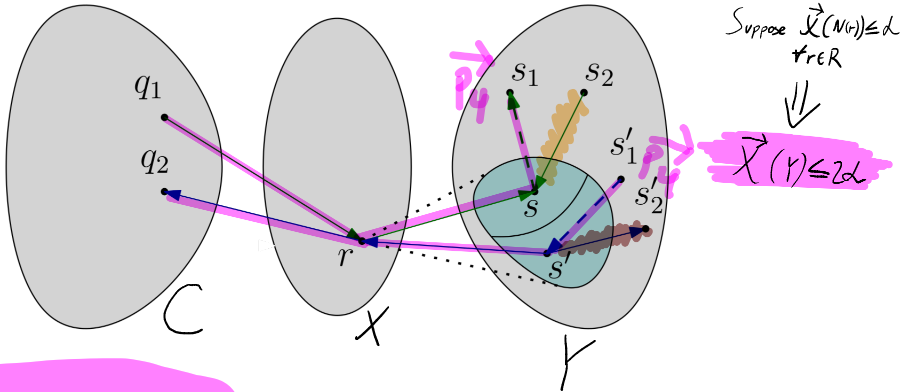

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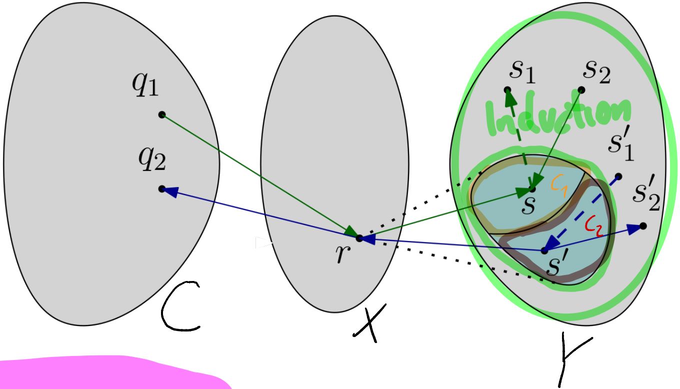


Suppose $\vec{\chi}(N(r)) \leq \alpha$ $\forall r \in R$

$\Downarrow$

$\vec{\chi}(Y) \leq 2\alpha$

Show $\vec{\chi}(C \cup X \cup Z \cup Y) \leq c$

Proof sketch—Finding a dipolar set and the strongly connected case

If G contains strongly connected tournament of size $\omega(G)$:

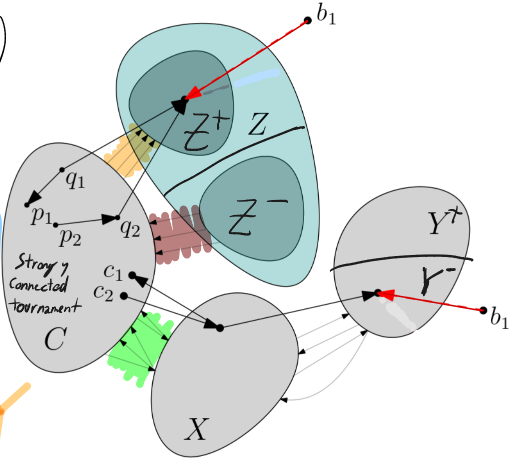
$\vec{\chi}(G) \leq f(\omega(G))$

Induction

$f(\omega(G)) \leq N(C)$

$\omega(G) f(\omega(G)-1)$

$+ 2f(\omega(G)-1)$ Y



$C \cup X \cup Z \cup Y$ is a dipolar set!

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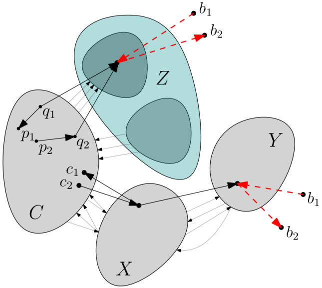

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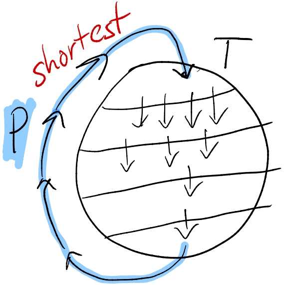
If G **does not** contain a strongly connected tournament of size $\omega(G)$:

Consider a tournament T together with a shortest path P making it strongly conneceted.

Take the one where P is the shortest.



Proof sketch—A hint for the other case

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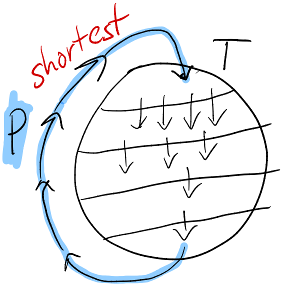

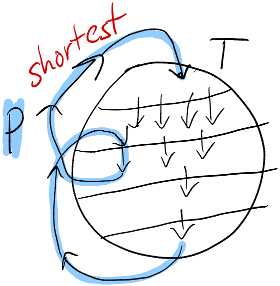

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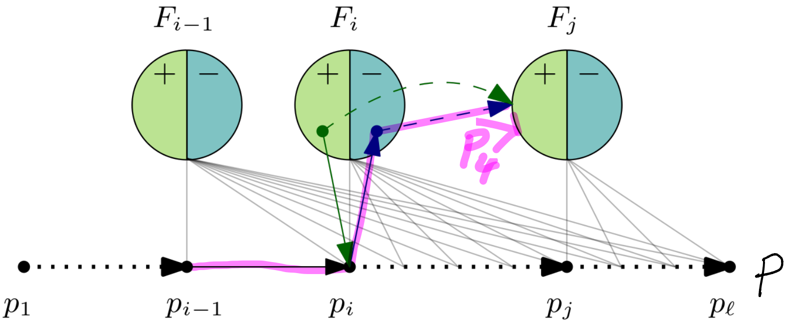

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Polynomial $\overrightarrow{\chi}$-boundedness

Is any oriented P_4 polynomially $\overrightarrow{\chi}$-bounding?

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Next step

Is oriented P_5 $\overrightarrow{\chi}$-bounding?

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Thank you!