List 3-Colouring is polynomial on $(P_2 + P_5)$-Free and $(P_3 + P_4)$-Free Graphs.

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ISAAC 2018,
Jiaoxi, Taiwan

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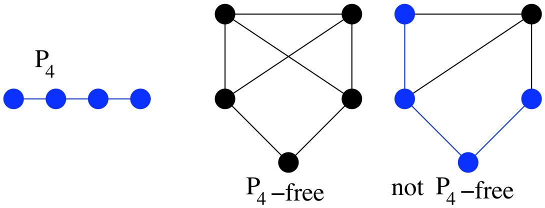

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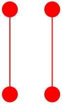
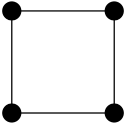
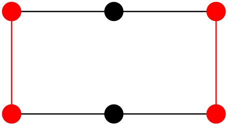


$2P_2$



$2P_2$-free



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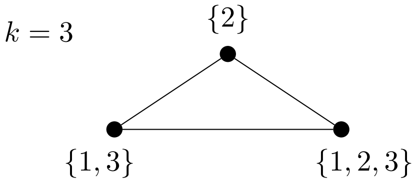

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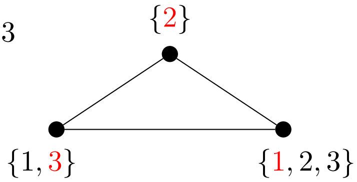

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- k-List Colouring: $\forall v \in V : |L(v)| \leq k$.

State of the art (k-Colouring on H-free graphs)

- For every $k \geq 3$, k-Colouring on H-free graphs is NP-complete if H contains
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- Hence, it remains to consider the case where H is a linear forest

State of the art (P_t-free graphs)

	k-Colouring			
t	$k=3$	$k=4$	$k=5$	$k\geq 6$
$t\leq 5$	P	P	P	P
$t=6$	P	P	NP-c	NP-c
$t=7$	P	NP-c	NP-c	NP-c
$t\geq 8$	?	NP-c	NP-c	NP-c

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State of the art (linear forests)

Theorem [survey: Golovach, Johnson, Paulusma, Song 17]

For a graph H with $|V(H)| \leq 6$, 3-Colouring and List 3-Colouring are polynomial-time solvable on H-free graphs if and only if H is a linear forest.

D. Paulusma, C&C 2017

3-Coloring for H-free graphs is classified if $|V(H)| \leq 7$ except when

- $H = P_2 + P_5$
- $H = P_3 + P_4$.

Our results

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Theorem (Chudnovsky, Huang, Spirkl, Zhong 2018+)

List 3-Colouring is polynomial-time solvable for $(kP_3 + P_6)$-free graphs.

Part of the proof

for $(P_2 + P_5)$-free and $(P_3 + P_4)$-free graphs

Simple example of our techniques

Theorem [Edwards 86]

2-List Colouring is polynomial-time solvable for general graphs.

- (i.e., every vertex is in D or in the neighbourhood of D)
- dominating: If we colour D, we reduce all lists of vertices in G at least by one (instance of 2-list coloring).
- constant size: we can try all possible colourings of $D \rightarrow$ we can examine all of them.

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Proof: Used tools

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Theorem [Bonomo, Chudnovsky, Maceli, Schaudt, Stein, Zhong 17]

List 3-Colouring is polynomial-time solvable for P_7-free graphs.

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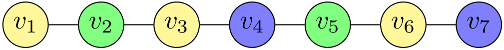

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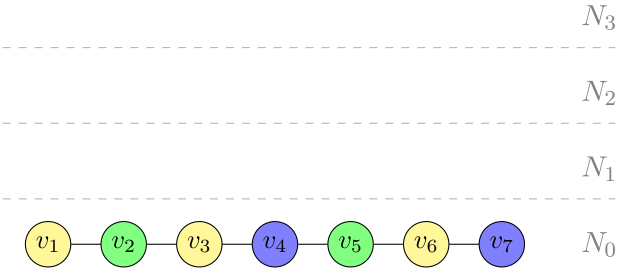

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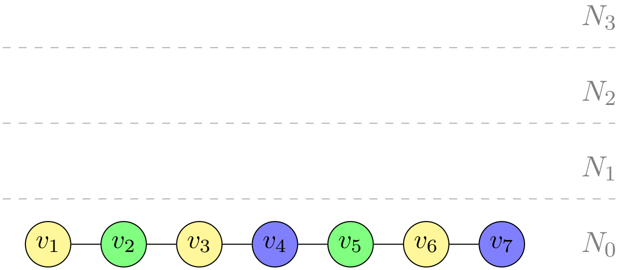
- Reduce to a polynomial number of instances of 2-List Colouring.



Overview of the polynomial algorithm

- Preprocessing
- Phase 1: Reduce private neighbours of non-adjacent vertices in P_7
- Phase 2: Two types of lists in the first layer
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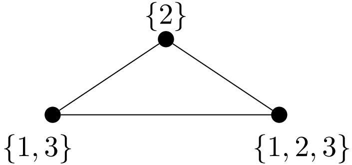

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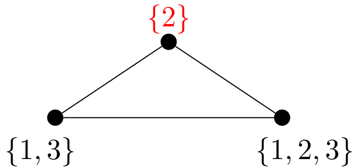

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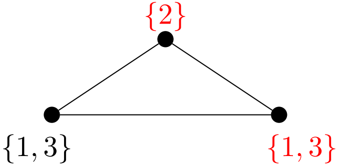

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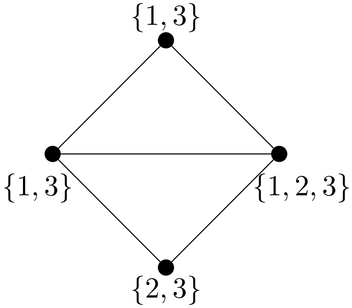
  Rule 6. (diamond colour propagation) If u and v are adjacent and share two common neighbours x and y with $L(x) \neq L(y)$, then set $L(x) := L(x) \cap L(y)$ and $L(y) := L(x) \cap L(y)$.



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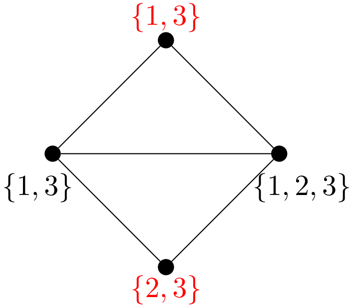

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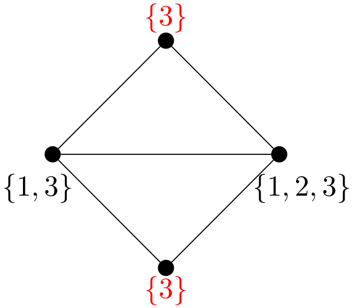

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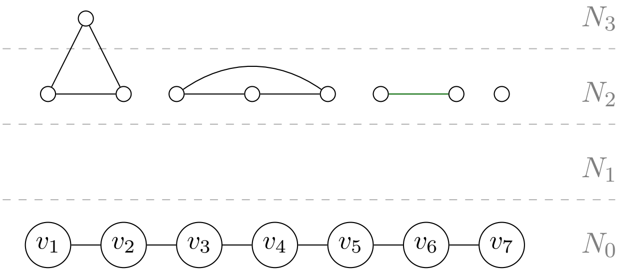
- We can enforce strong structure on $N_2 \cup N_3$:



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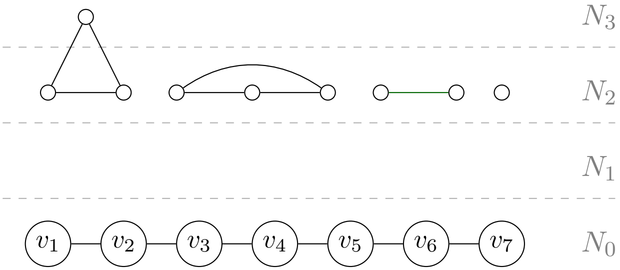

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We call a vertex u in N_2 and its neighbours in N_1 active if $|L(u)| = 3$.

- We use A_1, A_2 for the set of all active vertices in N_1, N_2.

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Goal

Deactivate all active vertices.

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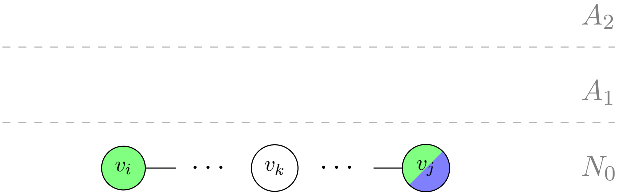
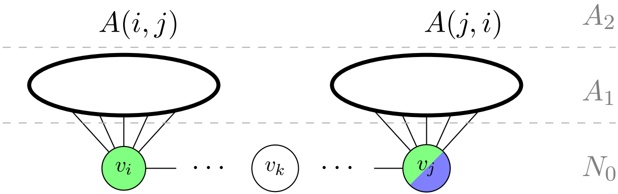
Phase I



Phase I

- $A(i,j)\ldots$ set of active vertices in N_1 adjacent to v_i and nonadjacent to v_j, similarly $A(j,i)$.



Phase I

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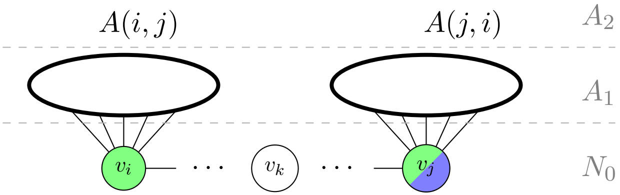
To deactivate at least one of $A(i,j), A(j,i)$ for nonadjacent v_i, v_j.



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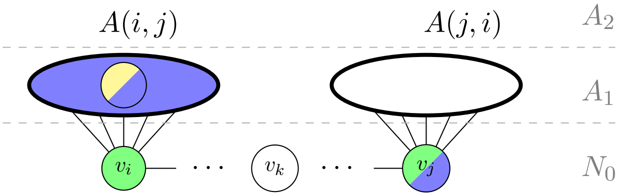
- Branching: at most one vertex in $A(i,j)$ is yellow $\rightarrow$ $n+1$ branches,



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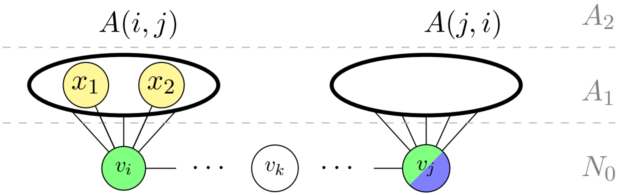

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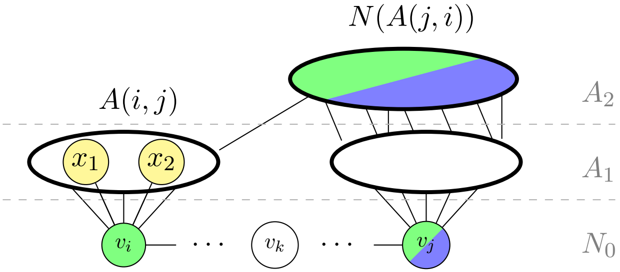

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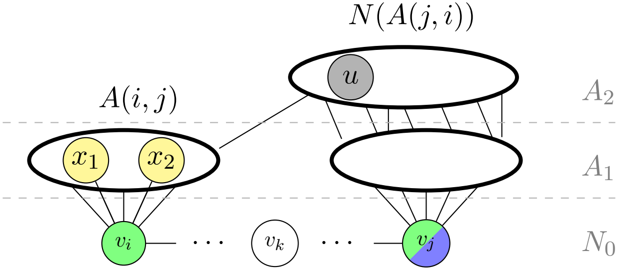

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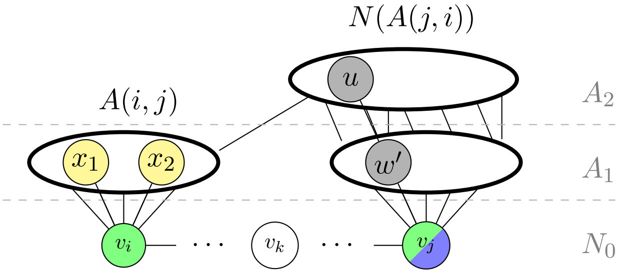
- Either $A(j,i)$ is deactivated, or we find an induced $P_5 + P_3$.



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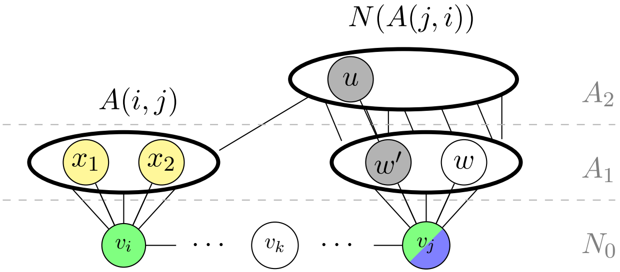

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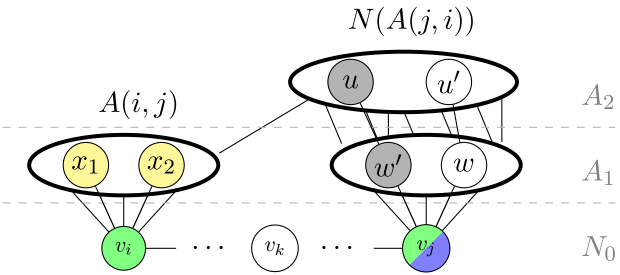

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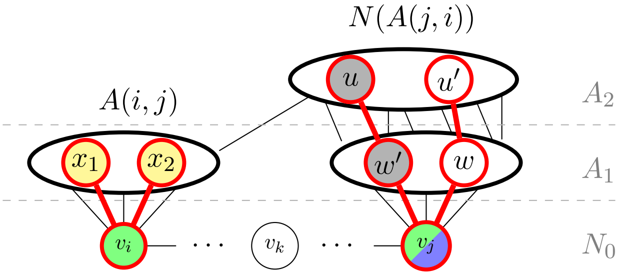

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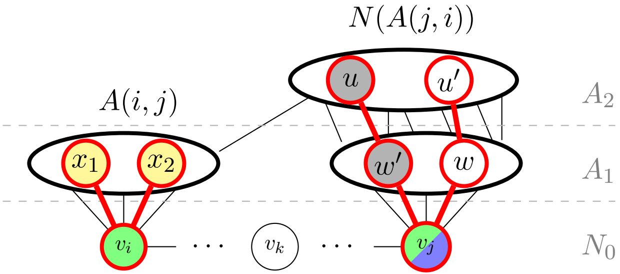
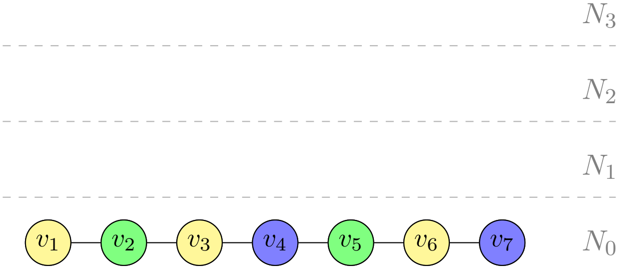
- $A(i,j)$ or $A(j,i)$ can be deactivated for all nonadjacent vertices v_i, $v_j \in N_0$



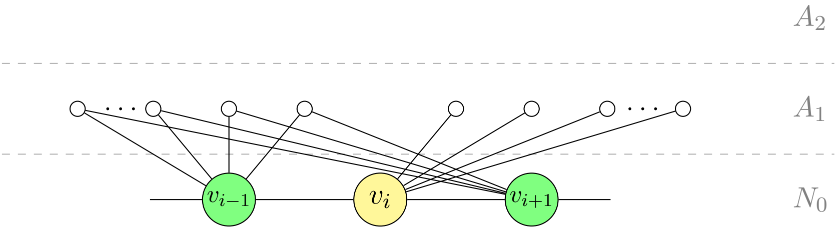
Phase I: Consequences

- $A(i, j)$ or $A(j, i)$ is empty for nonadjacent v_i, v_j
  $\rightarrow$ at most two types of lists of {●, ●}, {●, ●}, {●, ●} can occur in A_1,



Overview of the polynomial algorithm

- ✓ Preprocessing
- ✓ Phase 1: Reduce private neighbours of non-adjacent vertices in P_7
- Phase 2: Two types of lists in the first layer ({yellow, blue}, {blue, green})
- Phase 3: One type of lists in the first layer({blue, green})



Overview of the polynomial algorithm

- ✓ Preprocessing
- ✓ Phase 1: Reduce private neighbours of non-adjacent vertices in P_7
- ✓ Phase 2: Two types of lists in the first layer ({ , }, { , })
- Phase 3: One type of lists in the first layer ({ , })

- Colour all vertices in A_2 by the colour which is not in A_1.

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$(P_2 + P_5)$-free graphs

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Thank you for your attention!