Max Weight Independent Set in Graphs with no Long Claws: An Analog of the Gyárfás' Path Argument

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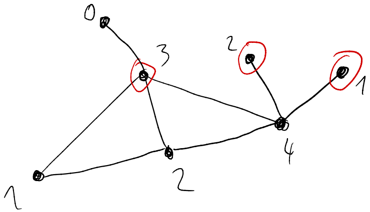
MIMUW Algorithms Seminar 2022

Max Weight Independent Set Problem

Definition (Max Weight Independent Set (MWIS))

Let G be a graph and let $\mathfrak{w} : V(G) \to \mathbb{R}$. The **MWIS** problem asks for a set $I \subseteq V(G)$ s.t. $G[I]$ is edgeless and $\mathfrak{w}(I)$ is as large as possible.



$\mathfrak{w}(I) = 6$

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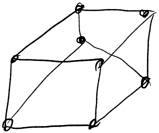
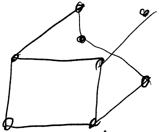
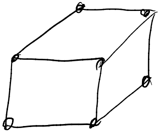


G



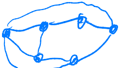
NOT Induced subgraph off G



Induced Subgraph off G

P4

P4-Free



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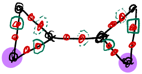

twice subdivided

For **one** forbidden subgraph H ('82 Alekseev):

- Subdividing strategy proves NP-completeness when H is **not a forest** or **have two degree-three vertices**. connected
- NP-complete when H does have **more than three leaves**.

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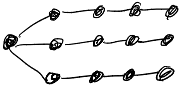
Let $S_{t,t,t}$ be a $t-1$ times subdivided claw.



$S_{4,4,4}$

Positive Results for MWIS

- **'19 Grzesik, Klimošová, Pilipczuk, Pilipczuk**
  $\rightsquigarrow$ Polynomial on P_6-free graphs
- **'20 Chudnovsky, Pilipczuk, Pilipczuk, Thomassé**
  $\rightsquigarrow$ QPTAS, subexp. on $S_{t,t,t}$-free graphs
- **'20 Gartland, Lokshtanov & '21 Pilipczuk, Pilipczuk, Rzążewski**
  $\rightsquigarrow$ Quasi-polynomial on P_t-free graphs
- **'21 Gartland, Lokshtanov, Pilipczuk, Pilipczuk, Rzążewski**
  $\rightsquigarrow$ Quasi-polynomial on $C_{\geq t}$-free graphs
- **'22 Abrishami, Chudnovsky, Dibek, and Rzążewski**
  $\rightsquigarrow$ Polynomial on $S_{t,t,t}$-free graphs of bounded degree

Gyárfás Path

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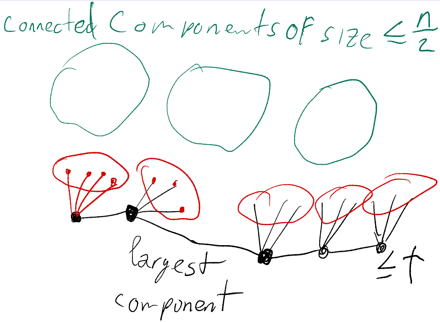


Theorem (Gyárfás '75)

*Given an n-vertex graph G, one can in polynomial time find an **induced path Q** in G such that every connected component of $G - N[V(Q)]$ has **at most $n/2$ vertices**.*

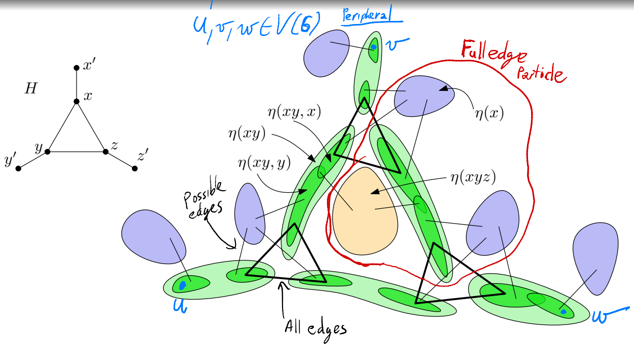
The Three-in-a-tree Theorem

Theorem ('10 Chudnovsky, Seymour)

*Let G be an n-vertex graph and consider $Z \subseteq V(G)$ **with** $|Z| \geq 2$. There is an algorithm that runs in time $\mathcal{O}(n^5)$ and returns one of the following:*

- *an induced subtree of G containing at least three elements of Z,*
- *an **e.s.d.** (H, η) of (G, Z).*

$u, v, w \in V(G)$

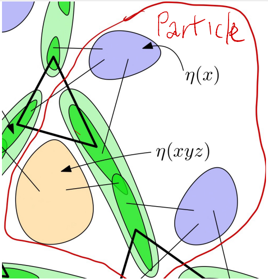


Our Result — Gyárfás' Path Analog for $S_{t,t,t}$-free Graphs

Theorem ('22 KM, TM, JN, KO, MP, PRz, MS)

Given an n-vertex graph G and $t \geq 1$, one can in polynomial time either:

- *output an induced copy of $S_{t,t,t}$ in G, or*
- *output a* ***set $\mathcal{P}$ consisting of at most $11 \log n + 6$ induced paths*** *in G, each of length at most $t+1$, and a rigid* ***e.s.d. of $G - N[\bigcup_{P \in \mathcal{P}} V(P)]$*** *whose every particle has at most $n/2$ vertices.*



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Applications

- **Subexponential algorithm:** in time exponential in $\mathcal{O}(\sqrt{n} \log n)$ for MWIS.
- **QPTAS:** in time exponential in $\mathcal{O}(\varepsilon^{-1} \log^5 n)$ we obtain $(1-\varepsilon)$-approximation for MWIS.

Proof

Lemma (Recursive formulation)

Fixed n

*Given: graph G; set $\mathcal{Q}$ **of at most two induced paths**; **refined e.s.d. of** $G - N[\bigcup \mathcal{Q}]$. In polynomial time, we output one of the following:*

- *an induced copy of $S_{t,t,t}$ in G, or*
- *$\mathcal{P}$, $X \subseteq N[\bigcup \mathcal{P}]$, and a **refined** e.s.d. $(G - X, \eta)$, so that $|\mathcal{P}| \leq 6 \log_{3/2}(|\bigcup \mathcal{Q}|) + 6$ and the longest path in $\mathcal{P}$ has at most $t + 1$ vertices.*

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a refined

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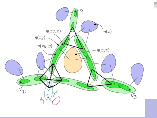
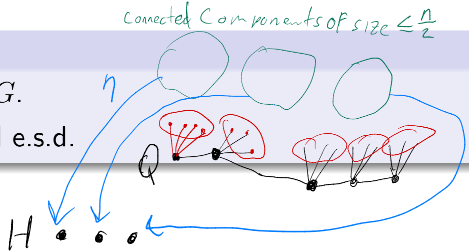
Initialization

- $\mathcal{Q}$ consist of Gyárfás path in G.
- Hence, $G - N[\bigcup \mathcal{Q}]$ is refined e.s.d.





Proof—First Base Case

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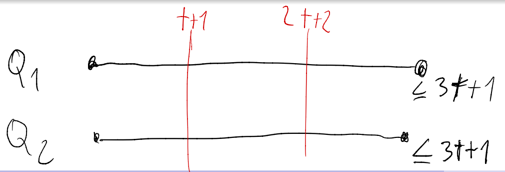
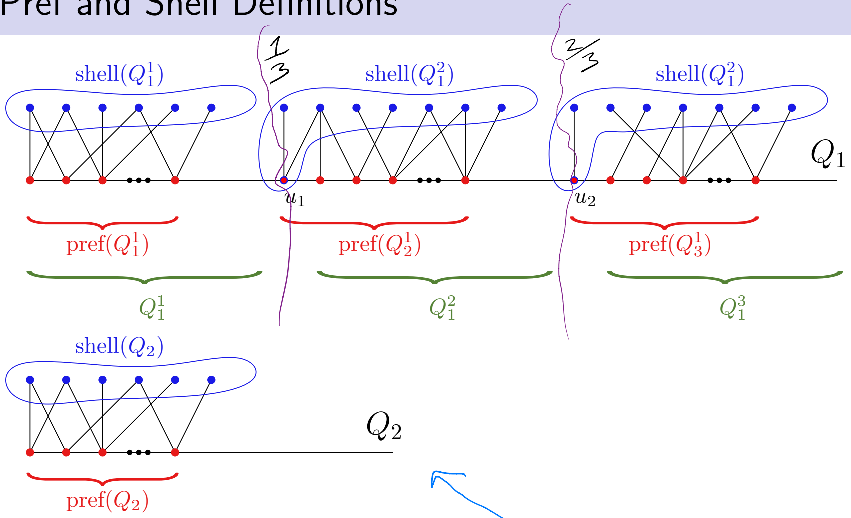
The longest path in $\mathcal{Q}$ has at most $3t + 1$ vertices.

We return $\mathcal{P} := \mathcal{Q}$ and e.s.d. given at the input. $X := N[\bigcup P]$



$\Longrightarrow$ ≤6 paths

Pref and Shell Definitions

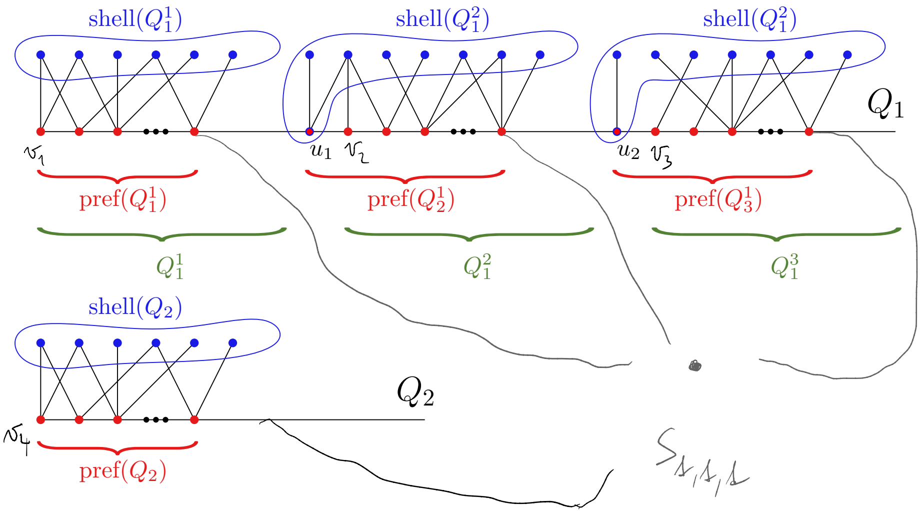


$\mathrm{shell}(P) := N[\mathrm{pref}(P)] \setminus \bigcup \mathcal{S}$ if $|P| \geq t$, $\quad \mathcal{S} := \{Q_1^1, Q_1^2, Q_1^3, Q_2\}$.

$\mathrm{shell}(P) := N[\mathrm{pref}(P)]$ otherwise.

Use of Tree-in-a-tree Theorem

$v_1, v_2, v_3, v_4 \in Z$



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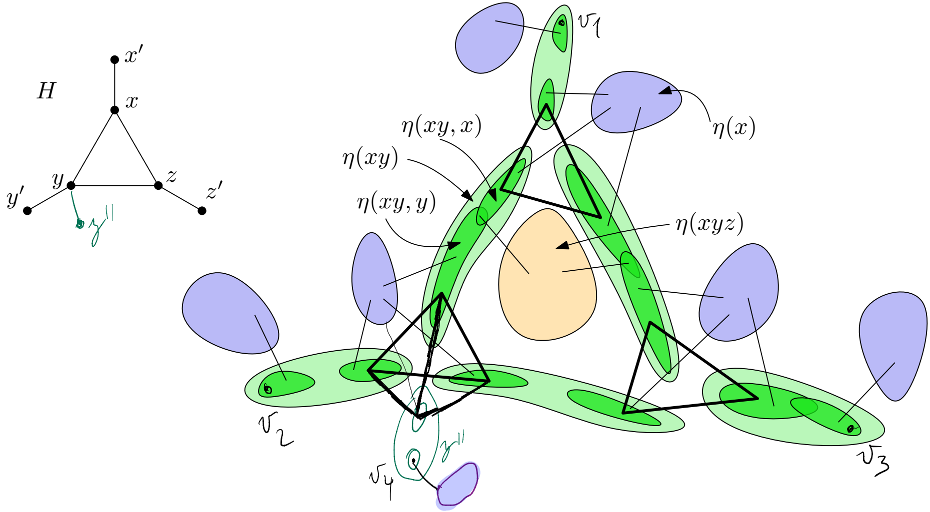

Proof—Second Base Case

Lemma (Recursive formulation)

*Given: graph G; set $\mathcal{Q}$ **of at most two induced paths**; **refined e.s.d. of** $G - N[\bigcup \mathcal{Q}]$. In polynomial time, we output one of the following:*

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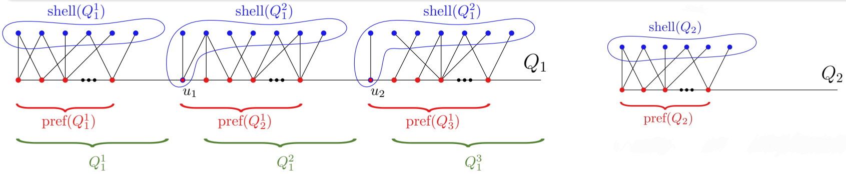
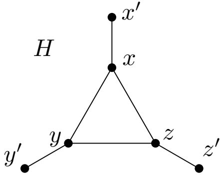
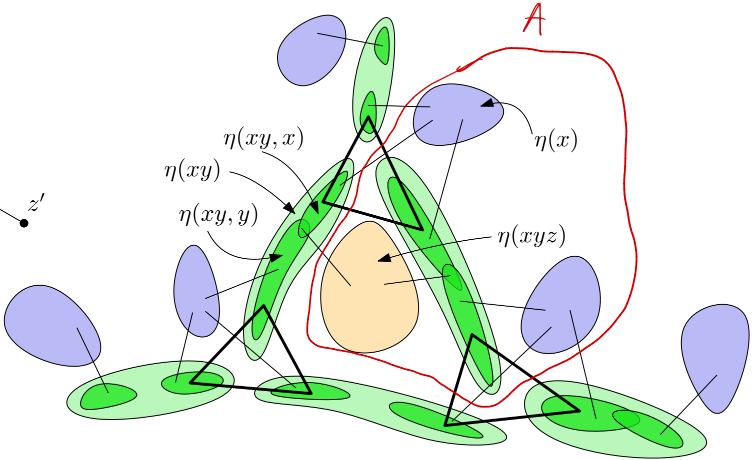
Three-in-a-tree theorem returned a refined e.s.d.

We return $\mathcal{P} := \mathrm{pref}(\mathcal{S})$, $X := \mathrm{shell}(\mathcal{S})$ and the refined e.s.d.



Tree-in-a-tree Theorem Retured Large Particle





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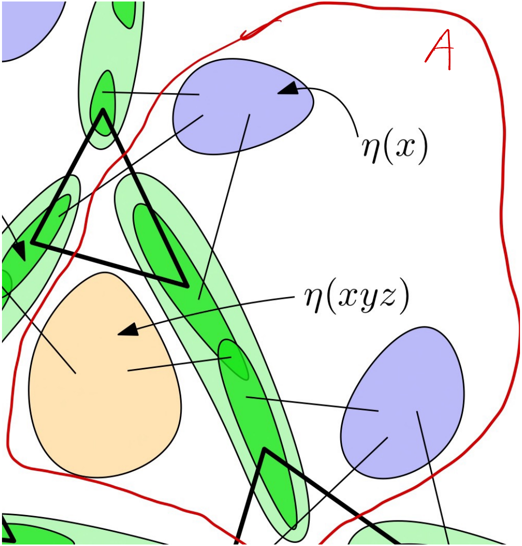

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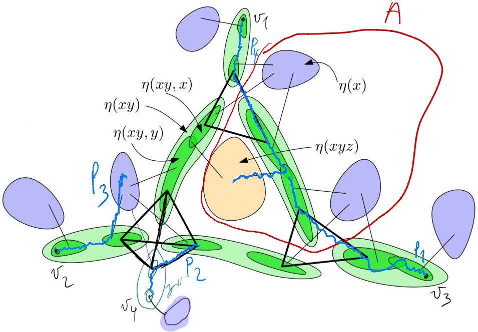

Theorem ('20 Chudnovsky, Pilipczuk, Pilipczuk, Thomassé)

*Let (H, η) be an e.s.d. of G. Suppose P_1, P_2, P_3 are three induced paths in G that do not **touch** each other, and moreover each of P_1, P_2, P_3 has an endvertex that is **peripheral** in (H, η). Then in (H, η) there is **NO** particle that touches each of P_1, P_2, P_3.*

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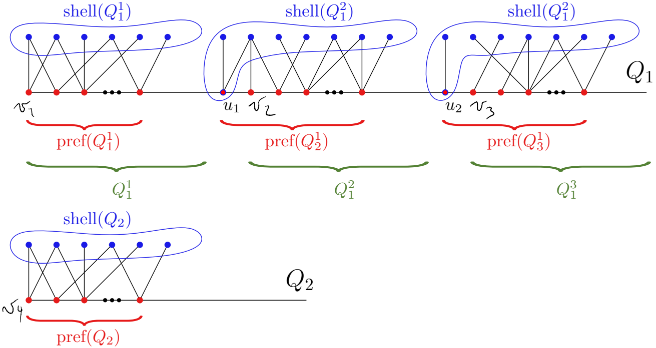
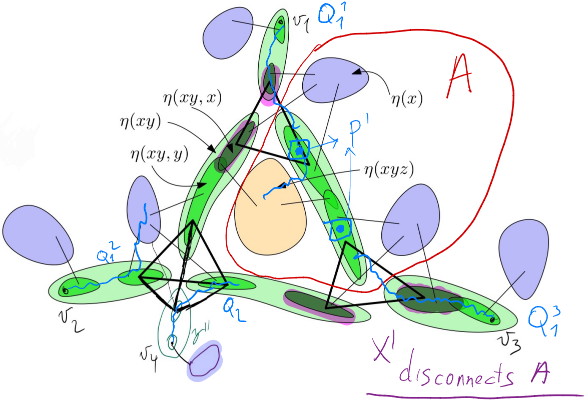


Proof—Recursive Step

We recurse on particle A with at most two touching paths.

- A can be separated by two vertices, denoted as $\mathcal{P}'$ ($X' := N[\bigcup \mathcal{P}'] - A$)
- $|\bigcup \mathcal{Q}|$ drops by at least $2/3$!



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Let $\hat{\mathcal{P}}, \hat{X}$ and refined e.s.d. be obtained from the recursion.

≤ 2 path $\quad \le 4$ paths

We return $\mathcal{P} := \hat{\mathcal{P}} \cup \mathcal{P}' \cup \mathrm{pref}(\mathcal{S})$, $X := \hat{X} \cup X' \cup \mathrm{shell}(\mathcal{S})$, ...

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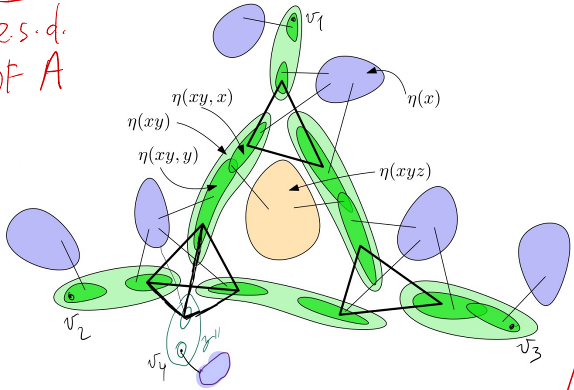
$G - A - X$ is cc

and has size $\leq \frac{n}{2}$

make it an isolated vertex bag

e.s.d. of A



Open Problems

Conjecture ('22 KM, TM, JN, KO, MP, PRz, MS)

*For every integer $t \geq 1$ there exists a constant $\varepsilon > 0$ and an integer s such that every $S_{t,t,t}$-free graph G
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