

# Parameterized Complexity of Fair Vertex Evaluation Problems

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## Abstract

A prototypical graph problem is centered around a graph-theoretic property for a set of vertices and a solution to it is a set of vertices for which the desired property holds. The task is to decide whether, in the given graph, there exists a solution of a certain quality, where we use size as a quality measure. In this work, we are changing the measure to the fair measure [Lin&Sahni: Fair edge deletion problems. IEEE Trans. Comput. 89]. The measure is  $k$  if the number of solution neighbors does not exceed  $k$  for any vertex in the graph. One possible way to study graph problems is by defining the property in a certain logic. For a given objective an evaluation problem is to find a set (of vertices) that simultaneously minimizes the assumed measure and satisfies an appropriate formula.

In the presented paper we show that there is an FPT algorithm for the MSO FAIR VERTEX EVALUATION problem for formulas with one free variable parameterized by the twin cover number of the input graph. Here, the free variable corresponds to the solution sought. One may define an extended variant of MSO FAIR VERTEX EVALUATION for formulas with  $\ell$  free variables; here we measure a maximum number of neighbors in each of the  $\ell$  sets. However, such variant is  $W[1]$ -hard for parameter  $\ell$  even on graphs with twin cover one. Furthermore, we study the FAIR VERTEX COVER (FAIR VC) problem. FAIR VC is among the simplest problems with respect to the demanded property (i.e., the rest forms an edgeless graph). On the negative side, FAIR VC is  $W[1]$ -hard when parameterized by both treedepth and feedback vertex set of the input graph. On the positive side, we provide an FPT algorithm for the parameter modular width.

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## 1 Introduction

A prototypical graph problem is centered around a fixed property for a set of vertices. A solution is any set of vertices for which the given property holds. In a similar manner, one can define the solution as a set of vertices such that the given property holds when we remove this set of vertices from the input graph. This leads to the introduction of deletion

problems—a standard reformulation of some classical problems in combinatorial optimization introduced by Yannakakis [42]. Formally, for a graph property  $\pi$  we formulate a *vertex deletion problem*. That is, given a graph  $G = (V, E)$ , find the smallest possible set of vertices  $W$  such that  $G \setminus W$  satisfies the property  $\pi$ . Many classical problems can be formulated in this way such as MINIMUM VERTEX COVER ( $\pi$  describes an edgeless graph) or MINIMUM FEEDBACK VERTEX SET ( $\pi$  is valid for forests).

Clearly, the complexity of a graph problem is governed by the associated predicate  $\pi$ . We can either study one particular problem or a broader class of problems with related graph-theoretic properties. One such relation comes from logic, for example, two properties are related if it is possible to express both by a first order (FO) formula. Then, it is possible to design a *model checking algorithm* that for any property  $\pi$  expressible in the fixed logic decides whether the input graph with the vertices from  $W$  removed satisfies  $\pi$  or not.

Undoubtedly, Courcelle’s Theorem [6] for graph properties expressible in the monadic second-order logic (MSO) on graphs of bounded treewidth plays a prime role among model checking algorithms. In particular, Courcelle’s Theorem provides for an MSO sentence  $\varphi$  an algorithm that given an  $n$ -vertex graph  $G$  with treewidth  $k$  decides whether  $\varphi$  holds in  $G$  in time  $f(k, |\varphi|)n$ , where  $f$  is some computable function and  $|\varphi|$  is the quantifier depth of  $\varphi$ . In terms of parameterized complexity, such an algorithm is called *fixed-parameter tractable* (the problem belongs to the class FPT for the combined parameter  $k + |\varphi|$ ). We refer the reader to monographs [9, 10] for background on parameterized complexity theory and algorithm design. There are many more FPT model-checking algorithms, e.g., an algorithm for (existential counting) modal logic model checking on graphs of bounded treewidth [38], MSO model checking on graphs of bounded neighborhood diversity [29], or MSO model checking on graphs of bounded shrubdepth [18] (generalizing the previous result). First order logic (FO) model checking received recently quite some attention as well and algorithms for graphs with bounded degree [39], nowhere dense graphs [21], and some dense graph classes [15] were given.

Kernelization is one of the most prominent techniques for designing FPT algorithms [11]. It is a preprocessing routine that in polynomial time reduces the input instance to an equivalent one whose size and parameter value can be upper-bounded in terms of the input (i.e., original) parameter value. It should be noted that kernelization is not really common for model checking algorithms.<sup>1</sup> There are few notable exceptions—the result of Lampis [29] (see Proposition 7) was, up to our knowledge, the first result of this kind. Lampis [29] presented a kernel for  $\text{MSO}_1$  model checking in graphs of bounded neighborhood diversity (of exponential size in quantifier depth). The aforementioned result has been recently followed by Gajarský and Hliněný [14] who showed a kernel for  $\text{MSO}_1$  model checking in graphs of bounded shrubdepth (a parameter generalizing neighborhood diversity, twin cover, and treedepth). It is worth noting that it was possible to use the kernel of [29] to extend his model checking algorithm for so-called fair objectives and their generalizations [34, 25]. In this work, we continue this line of research.

**Fair Objective.** Fair deletion problems, introduced by Lin and Sahni [33], are such modifications of deletion problems where instead of minimizing the size of the deleted set we change the objective. The FAIR VERTEX DELETION problem is defined as follows. For a given graph  $G = (V, E)$  and a property  $\pi$ , the task is to find a set  $W \subseteq V$  which minimizes the maximum number of neighbors in the set  $W$  over all vertices, such that the property  $\pi$

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<sup>1</sup> We are aware of the equivalence of FPT and kernels (cf. [9]), however, we would like to point out that the difference between a directly applicable set of rules and a theoretical bound is large.

holds in  $G \setminus W$ . Intuitively, the solution should not be concentrated in a neighborhood of any vertex. In order to classify (fair) vertex deletion problems we study the associated decision version, that is, we are interested in finding a set  $W$  of size at most  $k$ , for a given number  $k$ . Note that this can introduce only polynomial slowdown, as the value of our objective is bounded by 0 from below and by the number of vertices of the input graph from above. Since we are about to use a formula with a free variable  $X$  to express the desired property  $\pi$ , we naturally use  $X$  to represent the set of deleted vertices in the formula. The *fair cost* of a set  $W \subseteq V$  is defined as  $\max_{v \in V} |N(v) \cap W|$ . We refer to the function that assigns each set  $W \subseteq V$  its fair cost as to the *fair objective function*. Here, we give a formal definition of the computational problem when the property  $\pi$  is defined in some logic L.

**FAIR L VERTEX DELETION**

**Input:** An undirected graph  $G = (V, E)$  and a positive integer  $k$ , an L sentence  $\varphi$ .  
**Question:** Is there a set  $W \subseteq V$  of fair cost at most  $k$  such that  $G \setminus W \models \varphi$ ?

Let  $\varphi(X)$  be a formula with one free variable and let  $G = (V, E)$  be a graph. For a set  $W \subseteq V$  by  $\varphi(W)$  we mean that we substitute  $W$  for  $X$  in  $\varphi$ . The definition bellow can be naturally generalized to contain  $\ell$  free variables. We would like to point out one crucial difference between deletion and evaluation problems, namely that in evaluation problems we have access to the variable that represents the solution. This enables evaluation problems to impose some conditions on the solution, e.g., we can ensure that the graph induced by the solution has diameter at most 2 or is triangle-free.

**FAIR L VERTEX EVALUATION<sup>2</sup>**

**Input:** An undirected graph  $G = (V, E)$  and a positive integer  $k$ , an L formula  $\varphi(X)$  with one free variable.  
**Question:** Is there a set  $W \subseteq V$  of fair cost at most  $k$  such that  $G \models \varphi(W)$ ?

One can define *edge deletion problems* in a similar way as vertex deletion problems.

**FAIR L EDGE DELETION**

**Input:** An undirected graph  $G = (V, E)$ , an L sentence  $\varphi$ , and a positive integer  $k$ .  
**Question:** Is there a set  $F \subseteq E$  such that  $G \setminus F \models \varphi$  and for every vertex  $v$  of  $G$ , it holds that  $|\{e \in F : e \ni v\}| \leq k$ ?

The evaluation variant is analogical. Recall, in dense graph classes one cannot obtain an  $\text{MSO}_2$  model checking algorithm running in FPT-time [30]. That is a reason why evaluation problems does not make sense in this context. In sparse graph classes, this problem was studied in [34] where  $\text{W}[1]$ -hardness was obtained for Fair FO Edge Deletion on graphs of bounded treedepth and FPT algorithm was derived for Fair  $\text{MSO}_2$  Edge Evaluation on graphs of bounded vertex cover.

Minimizing the fair cost arises naturally for edge problems in many situations as well, e.g., in defective coloring [8], which has been substantially studied from the practical network communication point of view [22, 24]. A graph  $G = (V, E)$  is  $(k, d)$ -colorable if every vertex can be assigned a color from the set  $[k]$  in such a way that every vertex has at most  $d$  neighbors of the same color. This problem can be reformulated in terms of fair deletion; we aim to find a set of edges  $F$  such that graph  $G' = (V, F)$  has maximum degree  $d$  and  $G \setminus F$  can be partitioned into  $k$  independent sets.

<sup>2</sup> This problem is called GENERALIZED FAIR L VERTEX-DELETION in [34]. However, we believe that evaluation is a more suitable expression and coincides with standard terminology in logic.

**Related Results.** There are several possible research directions once a model checking algorithm is known. One possibility is to broaden the result either by enlarging the class of graphs it works for or by extending the expressive power of the concerned logic, e.g., by introducing a new predicate [28]. Another obvious possibility is to find the exact complexity of the general model checking problem by providing better algorithms (e.g., for subclasses [29]) and/or lower-bounds for the problem [13, 30]. Finally, one may instead of deciding a sentence turn attention to finding a set satisfying a formula with a free variable that is optimal with respect to some objective function [1, 7, 19]. In this work, we take the last presented approach—extending a previous work on MSO model checking for the fair objective function.

When extending a model checking result to incorporate an objective function or a predicate, we face two substantial difficulties. On the one hand, we are trying to introduce as strong extension as possible, while on the other, we try not to worsen the running time too much. Of course, these two are in a clash. One evident possibility is to sacrifice the running time and obtain an XP algorithm, that is an algorithm running in time  $f(k)|G|^{g(k)}$ . As an example there is an XP algorithm for the FAIR MSO<sub>2</sub> VERTEX EVALUATION problems parameterized by the treewidth (and the formula) by Kolman et al. [26] running in time  $f(|\varphi|, \text{tw}(G))|G|^{g(\text{tw}(G))}$ . An orthogonal extension is due to Szeider [40] for the so-called local cardinality constraints (MSO-LCC) who gave an XP algorithm parameterized by treewidth. If we decide to keep the FPT running time, such result is not possible for treedepth (consequently for treewidth) as we give W[1]-hardness result for a very basic FAIR L<sub>0</sub> VERTEX DELETION problem<sup>3</sup> in this paper. A weaker form of this hardness was already known for FO logic [34]. However, Ganian and Obdržálek [19] study CardMSO and provide an FPT algorithm parameterized by neighborhood diversity. Recently, Masařík and Toufar [34] examined fair objective and provide an FPT algorithm for the FAIR MSO<sub>1</sub> VERTEX EVALUATION problem parameterized by neighborhood diversity. See also [25] for a comprehensive discussion of various extensions of the MSO.

Since classical Courcelle’s theorem [6] have an exponential tower dependence on the quantifier depth of the MSO formula, it could be interesting to find a more efficient algorithm. However, Frick and Grohe [13] proved that this dependence is unavoidable, unless  $P = NP$ . On the other hand, there are classes where MSO model checking can be done in asymptotically faster time, e.g., the neighborhood diversity admits a single-exponential dependence on the quantifier depth of the formula [29]. For a recent survey of algorithmic metatheorems consult the article by Grohe and Kreutzer [20].

We want to turn a particular attention to the FAIR VERTEX COVER (FAIR VC) problem which, besides its natural connection with VERTEX COVER, has some further interesting properties. For example, we can think about classical vertex cover as having several crossroads (vertices) and roads (edges) that we need to monitor. However, people could get uneasy if they will see too many policemen from one single crossroad. In contrast, if the vertex cover has low fair cost, it covers all roads while keeping a low number of policemen in the neighborhood of every single crossroad.

## 1.1 Our Results

We give a metatheorem for graphs having bounded twin cover. Twin cover (introduced by Ganian [16]; see also [17]) is one possible generalization of the vertex cover number. Here, we measure the number of vertices needed to cover all edges that are not twin-edges; an edge

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<sup>3</sup> Here, L<sub>0</sub> stands for any logic that can express an edgeless graph.

$\{u, v\}$  is a *twin-edge* if  $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ . Ganian introduced twin cover in the hope that it should be possible to extend algorithms designed for parameterization by the vertex cover number to a broader class of graphs.

► **Theorem 1.** *The FAIR MSO<sub>1</sub> VERTEX EVALUATION problem parameterized by the twin cover number and the quantifier depth of the formula admits an FPT algorithm.*

We want to point out here that in order to obtain this result we have to reprove the original result of Ganian [16] for MSO<sub>1</sub> model checking on graphs of bounded twin cover. For this, we extend arguments given by Lampis [30] in the design of an FPT algorithm for MSO<sub>1</sub> model checking on graphs of bounded neighborhood diversity. We do this to obtain better insight into the structure of the graph (a kernel) on which model checking is performed (its size is bounded by a function of the parameter). This, in turn, allows us to find a solution minimizing the fair cost and satisfying the MSO<sub>1</sub> formula. The result by Ganian in version [16] is based on the fact that graphs of bounded twin cover have bounded shrubdepth and so MSO<sub>1</sub> model checking algorithm on shrubdepth ([18, 14]) can be used.

When proving hardness results it is convenient to show the hardness result for a concrete problem that is expressible by an MSO<sub>1</sub> formula, yet as simple as possible. Therefore, we introduce a key problem for Fair Vertex Deletion—the FAIR VC problem.

FAIR VERTEX COVER (FAIR VC)  
**Input:** An undirected graph  $G = (V, E)$ , and a positive integer  $k$ .  
**Question:** Is there a set  $W \subseteq V$  of fair cost at most  $k$  such that  $G \setminus W$  is an independent set?

Fair VC problem can be expressed in any logic that can express an edgeless graph (we denote such logic  $L_\emptyset$ ) which is way weaker even than FO. Therefore, we propose a systematic study of FAIR VC problem which, up to our knowledge, have not been considered before.

► **Theorem 2.** *The FAIR VC problem parameterized by treedepth  $\text{td}(G)$  and feedback vertex set  $\text{fvs}(G)$  combined is  $W[1]$ -hard. Moreover, if there exists an algorithm running in time  $f(w)n^{o(\sqrt{w})}$ , where  $w := \text{td}(G) + \text{fvs}(G)$ , then Exponential Time Hypothesis fails.*

Note that this immediately yields  $W[1]$ -hardness and  $f(w)n^{o(\sqrt{w})}$  lower bound for FAIR  $L_\emptyset$  VERTEX EVALUATION. Previously, an  $f(w)n^{o(w^{1/3})}$  lower bound was given for FO logic by Masařík and Toufar [34]. Thus our result is stronger in both directions, i.e., the lower bound is stronger, and the logic is less powerful. On the other hand, we show that FAIR VC can be solved efficiently in dense graph models.

► **Theorem 3.** *The FAIR VC problem parameterized by modular width  $\text{mw}(G)$  admits an FPT algorithm with running time  $2^{\text{mw}(G)} \text{mw}(G)n^3$ , where  $n$  is the number of vertices in  $G$ .*

We point out that the FAIR VC problem is (rather trivially) AND-compositional and thus it does not admit a polynomial kernel for parameterization by modular width.

► **Lemma 4.** *The FAIR VC problem parameterized by the modular width of the input graph does not admit a polynomial kernel, unless  $\text{NP} \subseteq \text{coNP}/\text{poly}$ .*

Moreover, an analog to Theorem 3 cannot hold for parameterization by shrubdepth of the input graph. This is a consequence of Theorem 2 and [18, Proposition 3.4].

Another limitation in a rush for extensions of Theorem 1 is given when aiming for more free variables.

|             | vc                    | fvs + td         | tc                  | nd                    | cvd                | mw                 |
|-------------|-----------------------|------------------|---------------------|-----------------------|--------------------|--------------------|
| FAIR VC     | *                     | T2               | ✓                   | *                     | ?                  | T3                 |
| FV L DEL    | MSO <sub>2</sub> *    | L <sub>∅</sub> ✓ | MSO <sub>1</sub> ✓  | MSO <sub>1</sub> *    | ?                  | ?                  |
| FV L EVAL   | MSO <sub>2</sub> [34] | L <sub>∅</sub> ✓ | MSO <sub>1</sub> T1 | MSO <sub>1</sub> *    | ?                  | ?                  |
| ℓ-FV L EVAL | MSO <sub>1</sub> *    | L <sub>∅</sub> ✓ | MSO <sub>1</sub> T5 | MSO <sub>1</sub> [25] | MSO <sub>1</sub> ✓ | MSO <sub>1</sub> ✓ |
| FE L DEL    | MSO <sub>2</sub> [34] | FO [34]          | ?                   | ?                     | FO T6              | ?                  |

■ **Table 1** The table summarize some related (with a citation) and all the presented (with a reference) results on the studied parameters. Green cells denote FPT results, and red cells represent hardness results. Logic L in metatheorems is specified by a logic used in the respective theorem. Symbol \* denotes implied results from previous research and symbol ✓ denotes new implied results. A question mark (?) indicates that the complexity is unknown. The FAIR EDGE L DELETION problem is delimited from Vertex problems since there are no apparent relations between them.

#### ℓ-FAIR L VERTEX EVALUATION

**Input:** An undirected graph  $G = (V, E)$  and a positive integer  $k$ , an L formula  $\varphi(X_1, \dots, X_\ell)$  with  $\ell$  free variables.  
**Question:** Are there sets  $W_1, \dots, W_\ell \subseteq V$  such that  $G \models \varphi(W_1, \dots, W_\ell)$  having fair cost at most  $k$ ?

The *fair cost* of  $W_1, \dots, W_\ell$  is defined as  $\max_{v \in V} \max_{i \in [\ell]} |N(v) \cap W_i|$ . It is very surprising that such a generalization is not possible for parameterization by twin cover, since the same extension is possible for parameterization by neighborhood diversity [25]. In fact, they prove something even stronger, i.e., an FPT algorithm parameterized by neighborhood diversity in the context of  $\text{MSO}_{\text{fin}}^L$  is given in [25]. In  $\text{MSO}_{\text{fin}}^L$  one can specify both lower- and upper-bound for each vertex and each free variable (i.e., a feasibility interval is given for every vertex).

► **Theorem 5.** *The ℓ-FAIR FO VERTEX EVALUATION problem is W[1]-hard for parameter ℓ even on graphs with twin cover of size one. Moreover, if there exists an algorithm running in time  $f(\ell)n^{O(\ell/\log \ell)}$ , then Exponential Time Hypothesis fails.*

Furthemore, we obtain a hardness result for the Fair FO Edge Deletion problem parameterized by the cluster vertex deletion number. Observe that for any graph its cluster vertex deletion number is upper-bounded by the size of its twin cover.

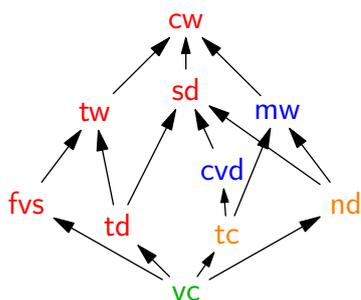
► **Theorem 6.** *The FAIR FO EDGE DELETION problem is W[1]-hard when parameterized by the cluster vertex deletion number of the input graph.*

For an overview of the results, please refer to Table 1 and to Figure 1 for the hierarchy of classes.

## 1.2 Preliminaries

We denote the set  $\{1, \dots, n\}$  by  $[n]$ . We deal with simple undirected graphs, for further standard notation we refer to monographs: graph theory [35] and parameterized complexity [9]. For a vertex  $v$  by  $N(v)$  we denote the neighborhood of  $v$  and by  $N[v]$  we denote the closed neighborhood of vertex  $v$ , i.e.,  $N(v) \cup \{v\}$ .

A parameter closely related to twin cover is *cluster vertex deletion* ( $\text{cvd}(G)$ ), that is, the smallest number of vertices one has to delete from a graph in order to get a collection of (disjoint) cliques. *Treedepth* of a graph  $G$  ( $\text{td}(G)$ ) is the minimum height of a rooted



■ **Figure 1** Hierarchy of graph parameters with depicted complexity of the FAIR L VERTEX EVALUATION problem. An arrow indicates that a graph parameter upper-bounds the other. Thus, hardness results are implied in the direction of arrows, and FPT algorithms are implied in the reverse direction. Green colors indicate FPT results for  $\text{MSO}_2$ , orange are FPT for  $\text{MSO}_1$ , blue are open, and red are hardness results. We denote treewidth by  $\text{tw}$ , shrubdepth by  $\text{sd}$ , and clique width by  $\text{cw}$ . We refer to book [9] for definitions. Other parameters and their respective abbreviations are defined in Subsection 1.2.

forest whose transitive closure contains the graph  $G$  [36]. *Feedback vertex set* ( $\text{fvs}(G)$ ) is the minimum number of vertices of a graph  $G$  whose removal leaves a graph without cycles. *Neighborhood diversity* ( $\text{nd}(G)$ ) is the smallest integer  $r$  such that the graph can be partitioned into  $r$  sets where each set is either complete graph or independent set and each pair of sets forms either a complete bipartite graph or there is no edge between them. *Modular width* of a graph  $G$  ( $\text{mw}(G)$ ), is the smallest positive integer  $r$  such that  $G$  can be obtained from an algebraic expression of width at most  $r$ , defined as follows. The *width of an expression*  $A$  is the maximum number of operands used by any occurrence of the substitution operation in  $A$ , where  $A$  is an algebraic expression that uses the following operations:

1. Create an isolated vertex.
2. The *substitution operation* with respect to a template graph  $T$  with vertex set  $[r]$  and graphs  $G_1, \dots, G_r$  created by algebraic expression. The substitution operation, denoted by  $T(G_1, \dots, G_r)$ , results in the graph on vertex set  $V = V_1 \cup \dots \cup V_r$  and edge set  $E = E_1 \cup \dots \cup E_r \cup \bigcup_{\{i,j\} \in E(T)} \{u, v\} : u \in V_i, v \in V_j\}$ , where  $G_i = (V_i, E_i)$  for all  $i \in [r]$ .

An algebraic expression of width  $\text{mw}(G)$  can be computed in linear time [41].

We stick with standard definitions and notation in logic. For a comprehensive summary, please consult a book by Libkin [32].

## 2 Metatheorems for Fair Evaluation

First we show an FPT algorithm for the FAIR  $\text{MSO}_1$  VERTEX EVALUATION problem parameterized by the twin cover number as it is stated in Theorem 1. We give a more detailed statement that implies the promised result straightforwardly. We split the proof into two parts. First, we show an algorithm for  $\text{MSO}_1$  model checking parameterized by twin cover of the graph (Proposition 9). In the second part, we prove that we can even compute the optimal fair cost (Proposition 13) and so derive the promised result.

**Overview of the Algorithm.** For the model checking algorithm, we closely follow the approach of Lampis [30]. The idea is that if there is a large set of vertices with the same closed neighborhood, then some of them are irrelevant, i.e., we can delete them without affecting the truthfulness of the given formula  $\varphi$ . For graphs of bounded neighborhood diversity using this rule alone can reduce the number of vertices below a bound that depends on  $\text{nd}(G)$  and  $|\varphi|$  only, thus providing an FPT model checking algorithm. For the graphs of bounded twin cover, this approach can be used to reduce the size of all (twin) cliques, yet their number can still be large. We take the approach one step further and describe the deletion of irrelevant cliques in a similar manner; these rules together yield a model checking algorithm for graphs of bounded twin cover.

The reduction rules also lead to a notion of *shape* of a set  $W \subseteq V$ . The motivation behind shapes is to partition all subsets of  $V$  such that if two sets  $W, W'$  have the same shape, then  $G \models \varphi(W)$  if and only if  $G \models \varphi(W')$ . This allows us to consider only one set of each shape for the purposes of model checking. Since the number of all distinct shapes is bounded by some function of parameters, we can essentially brute force through all possible shapes.

A final ingredient is an algorithm that for a given shape outputs a subset of vertices with this shape that minimizes the fair cost. This algorithm uses ILP techniques, in particular minimizing quasiconvex function subject to linear constraints.

**Notation.** In what follows  $G = (V, E)$  is a graph and  $K$  is its twin cover of size  $k$ . An  $\text{MSO}_1$  formula  $\varphi$  contains  $q_S$  set quantifiers and  $q_v$  element (vertex) quantifiers. Given a twin cover  $K$  and  $A \subseteq K$ , we say that  $A$  is the *cover set* of a set  $S \subseteq V \setminus K$  if every  $v \in S$  has  $N(v) \cap K = A$ . Note that, by the definition of twin cover, for all  $u, v \in V \setminus K$  with  $\{u, v\} \in E$  we have that  $A$  is a cover set for  $u$  if and only if  $A$  is a cover set for  $v$ . We say that two cliques have the same *type* if they have the same size and the same cover set. Clearly, if the cover set is fixed, two cliques agree on type if and only if their sizes are the same. We define a *labeled graph*, that is, a graph and a collection of labels on the vertices. We say that two cliques have the *same labeled type* if all of them have the same size, the same cover set and the same labels on vertices.

## 2.1 Model checking

We give a reformulated combination of Lemma 5 and Theorem 4 by Lampis [29].

► **Proposition 7** ([29, Lemma 5 and Theorem 4]). *Let  $\varphi$  be an  $\text{MSO}_1$  formula and let  $G$  be a labeled graph. If there is a set  $S$  of more than  $2^{q_S} q_v$  vertices having the same closed neighborhood and the same labels, then for any  $v \in S$  we have  $G \models \varphi$  if and only if  $G \setminus v \models \varphi$ .*

*In particular, if  $G$  is a graph with just one label, then for any clique  $C$  where each vertex has exactly the same closed neighborhood in  $G$  the following holds. Either there is a vertex  $v \in C$  such that  $G \models \varphi$  if and only if  $G \setminus v \models \varphi$  or the size of  $C$  is bounded by*

$$2^{q_S+1} q_v.$$

Proposition 7 bounds the size of a maximum clique in  $G \setminus K$  because we can apply it repeatedly for each clique that is bigger than the threshold  $2^{q_S+1} q_v$ . Now, we need to bound the number of cliques of each type. For this, we establish the following technical lemma.

► **Lemma 8.** *Let  $G$  be a labeled graph with twin cover  $K$ . Let  $\varphi$  be an  $\text{MSO}_1$  formula with  $q_v$  element quantifiers and  $q_S$  set quantifiers. Suppose the size of a maximum clique in  $G \setminus K$  is bounded by  $r$ . If there are strictly more than*

$$\alpha(q_S, q_v) = 2^{r q_S} (q_v + 1)$$

*cliques of the same labeled type  $\mathcal{T}$ , then there exists a clique  $C$  of the labeled type  $\mathcal{T}$  such that  $G \models \varphi$  if and only if  $G \setminus C \models \varphi$ .*

**Proof.** We prove the lemma by induction on  $q_S + q_v$ . Without loss of generality, all the quantifiers are assumed to be existential.

The base case of the induction is a quantifier-free formula. If there is at least one labeled type with at least two cliques  $C_1, C_2$  then the following holds. If  $G \models \varphi$  then  $G \setminus C_1 \models \varphi$  clearly holds as well since clique  $C_2$  has the same cover set and the same labels and a quantifier-free formula can only examine the labeled vertices. If  $(G \setminus C_1) \models \varphi$  and since  $C_1, C_2$  have the same labels the same size and the same cover set so  $G \models \varphi$ .

For the induction case, we consider the first quantified variable in  $\varphi$  and we split the proof whether it is set or vertex variable. Suppose there is at least one labeled type which contains strictly more than  $\alpha(q_S, q_v)$  cliques.

If it is a set variable then we try all possible assignments of the variable to cliques of the chosen labeled type. There are at most  $2^r$  of possible assignments to single clique and so from cliques of one labeled type emerge at most  $2^r$  different labeled (sub)types of cliques. We can compute that at least one of them has strictly more than  $\alpha(q_S - 1, q_v)$ :

$$\left\lceil \frac{\alpha(q_S, q_v) + 1}{2^r} \right\rceil \geq \left\lceil 2^{r(q_S - 1)}(q_v + 1) + \frac{1}{2^r} \right\rceil \geq \alpha(q_S - 1, q_v) + 1.$$

So, by the induction assumption we know that there is a clique  $C$  in the newly created labeled (sub)type of the promised properties and so of the larger labeled type.

If it is a vertex variable then only one more different labeled type can be created and importantly at most one single clique may contain the new label. We can compute:

$$\alpha(q_S, q_v) + 1 - 1 \geq 2^{r q_S} q_v + 2^{r q_S} \geq \alpha(q_S, q_v - 1) + 1.$$

The argument follows from the induction assumption by the same reasoning as in the previous case.  $\blacktriangleleft$

From this, we can derive a model checking algorithm.

► **Proposition 9** (Model checking on graphs of bounded twin cover). *Let  $G$  be a graph with twin cover  $K$  of size  $k$  and the size of the maximum clique in  $G \setminus K$  bounded by  $2^{q_S} q_v$  and  $\varphi$  is an  $\text{MSO}_1$  sentence then either there exists a clique  $C \in G \setminus K$  such that  $G \models \varphi$  if and only if  $G \setminus C \models \varphi$  or the size of  $G$  is bounded by*

$$k + (q_v + 1)q_v^2 2^{k+2q_S+2^{q_S} q_S q_v} = 2^{\mathcal{O}(k+2^{q_S} q_S q_v)}.$$

**Proof.** There are  $k$  vertices in the cover and  $2^k r$  types of cliques and each of them (by Lemma 8) is repeated at most  $\alpha(q_S, q_v) = 2^{r q_S} (q_v + 1)$  otherwise one clique of that type cannot be distinguished by formula  $\varphi$ . The maximal size of the clique is  $r = 2^{q_S} q_v$  from Proposition 7 and this gives us the desired bound.

$$\begin{aligned} k + 2^k r^2 \alpha(q_S, q_v) &= k + 2^k r^2 2^{r q_S} (q_v + 1) = k + 2^k (2^{q_S} q_v)^2 2^{2^{q_S} q_v q_S} (q_v + 1) = \\ &= k + (q_v + 1) q_v^2 2^{k+2q_S+2^{q_S} q_v q_S}. \end{aligned} \quad \blacktriangleleft$$

## 2.2 Finding a Fair Solution

In the upcoming proof we follow the ideas of Masařík and Toufar [34]. They define, for a given formula  $\varphi(X)$ , a so-called shape of a set  $W \subseteq V$  in  $G$ . The idea behind a shape is that in order to do the model checking we have deleted some vertices from  $G$  that cannot change the outcome of  $\varphi(X)$ , however, we have to derive a solution of minimal cost in the whole graph  $G$ . Thus the shape characterizes a set under which  $\varphi(X)$  holds and we have to be able to find a set  $W \subseteq V(G)$  for which  $\varphi(W)$  holds and  $W$  minimizes the fair cost among sets having this shape.

**Shape.** Let  $G = (V, E)$  be a graph,  $\varphi(X)$  an MSO formula,  $K \subseteq V$  a twin cover of  $G$ ,  $A \subseteq K$ , and let  $r = 2^{q_S+2} q_v$  and  $\alpha = 2^{r(q_S+1)}(q_v + 1)$ . Let  $\mathcal{C}$  be the collection of all cliques in  $G$  such that  $A$  is their cover set. We define an  $A$ -shape. An  $A$ -shape of size  $r$  is a two dimensional

|                           |          | number of vertices outside $W$ |   |   |   |   |   |          |
|---------------------------|----------|--------------------------------|---|---|---|---|---|----------|
|                           |          | 0                              | 1 | 2 | 3 | 4 | 5 | $\geq 6$ |
| number of vertices in $W$ | 0        |                                |   |   |   |   |   |          |
|                           | 1        |                                |   |   |   |   |   |          |
|                           | 2        |                                |   |   |   |   |   |          |
|                           | 3        |                                |   |   |   |   |   |          |
|                           | 4        |                                |   |   |   |   |   |          |
|                           | 5        |                                |   |   |   |   |   |          |
|                           | $\geq 6$ |                                |   |   |   |   |   |          |

■ **Figure 2** Example of a  $7 \times 7$   $A$ -shape. All cliques of size 3 will be assigned to yellow (light gray) fields, while cliques of size 8 will be assigned to orange (darker gray) fields.

|                           |          | number of vertices outside $W$ |   |   |   |   |   |          |
|---------------------------|----------|--------------------------------|---|---|---|---|---|----------|
|                           |          | 0                              | 1 | 2 | 3 | 4 | 5 | $\geq 6$ |
| number of vertices in $W$ | 0        | 0                              | 0 | 0 | 0 | 0 | 0 | 0        |
|                           | 1        | 1                              | 1 | 1 | 1 | 1 | 1 | 1        |
|                           | 2        | 2                              | 2 | 2 | 2 | 2 | 2 | 2        |
|                           | 3        | 3                              | 3 | 3 | 3 | 3 | 3 | 3        |
|                           | 4        | 4                              | 4 | 4 | 4 | 4 | 4 | 4        |
|                           | 5        | 5                              | 5 | 5 | 5 | 5 | 5 | 5        |
|                           | $\geq 6$ | ?                              | ? | ? | ? | ? | ? | 6        |

■ **Figure 3** An example of uncertainty in computation of objective function. The value in the last row depends on the size of the clique we are assigning to those cells. The value in the cell is how much we pay for any compatible clique assigned to this cell.

table  $S_A$  of dimension  $(r+2) \times (r+2)$  indexed by  $\{0, 1, \dots, r+1\} \times \{0, 1, \dots, r+1\}$ . Each entry  $S_A(i, j) \in \{0, \dots, \alpha+1\}$ . The interpretation of  $S_A(i, j)$  is the minimum of  $\alpha+1$  and of the number of cliques  $C$  with  $N(C) = A$  such that

$$\min(\alpha+1, |C \cap W|) = i \text{ and } \min(\alpha+1, |C \setminus W|) = j.$$

Finally, the shape of  $X$  in  $G$  is a collection of  $A$ -shapes for all  $A \subseteq K$ .

A solution for  $\mathcal{C}$  with cover set  $A$  can be formally described by a function  $\text{sol}: \mathcal{C} \rightarrow \mathbb{N} \times \mathbb{N}$ . The solution  $\text{sol}$  is *valid* if for every  $C \in \mathcal{C}$  with  $\text{sol}(C) = (i, j)$  either  $i+j = |C|$  or  $|C| \geq r$ ,  $i = r+1$  (or equivalently  $j = r+1$ ), and  $i+j < |C|$ . For an illustration of a valid assignment please refer to Figure 2. We say that a valid solution  $\text{sol}$  is *compatible* with the shape  $S_A$  if  $S(i, j) = |\text{sol}^{-1}(i, j)|$ , whenever  $S(i, j) \leq \alpha$  and  $|\text{sol}^{-1}(i, j)| \geq \alpha+1$  if  $S(i, j) = \alpha+1$ . The  $A$ -shape  $S_A$  is said to be *valid* if there exists a valid solution for  $S_A$ . Note that such a solution does not exist if the shape specifies too many (or too few) cliques of certain sizes. The shape  $S$  is *valid* if all its  $A$ -shapes are valid.

The following lemma is a key observation about shapes.

► **Lemma 10.** *Let  $\varphi$  be an  $\text{MSO}_1$  formula with one free variable,  $G$  a graph and  $W, W'$  two subsets of vertices having the same shape. Then  $G \models \varphi(W)$  if and only if  $G \models \varphi(W')$ .*

**Proof.** The proof follows using Proposition 7 and Lemma 8. Indeed, if we take the graph  $G$  with one label corresponding to set  $W$  and apply the reduction rules given by Proposition 7 and Lemma 8 and repeat the same process with  $W'$ , we obtain two isomorphic labeled graphs. ◀

Lemma 10 allows us to say that a formula with one free variable holds under a shape since it is irrelevant which subset of vertices of this particular shape is assigned to the free variable. Also note that deciding whether the formula holds under the shape can be done in FPT time by simply picking arbitrary assignment of the given shape and running a model checking algorithm. Lemma 12 computes a solution of minimal cost for an  $A$ -shape. We do this by reducing the task to integer linear programming (ILP) while using non-linear objective. A function  $f: \mathbb{R}^p \rightarrow \mathbb{R}$  is *separable convex* if there exist convex functions  $f_i: \mathbb{R} \rightarrow \mathbb{R}$  for  $i \in [p]$  such that  $f(x_1, \dots, x_p) = \sum_{i=1}^p f_i(x_i)$ .

► **Theorem 11** ([37] – simplified). *Integer linear programming with objective minimization of a separable convex function in dimension  $p$  is FPT with respect to  $p$  and space exponential in  $L$  the length of encoding of the ILP instance.*

► **Lemma 12.** *Let  $G = (V, E)$  be a graph,  $K$  be a twin cover of  $G$ , and  $\emptyset \neq A \subseteq K$ . There is an algorithm that given an  $A$ -shape  $S_A$  of size  $r$  computes a valid solution for  $S_A$  of minimal cost in time  $f(|K|, r) \cdot |G|^{O(1)}$  or reports that  $S_A$  is not valid.*

**Proof.** Let  $\mathcal{C}$  be the collection of all cliques such that  $A$  is their cover set. We split the task of finding a minimal solution to  $S_A$  into two independent parts depending on the size of cliques assigned in the phase.

The first phase is for cliques in  $\mathcal{C}$  with sizes at most  $r$ . Observe that these can be assigned deterministically in a greedy way. This is because no cell of  $S_A$  is shared by two sizes and we can see that if there are more cells with value  $\alpha$  on the corresponding diagonal we can always prefer the top one as this minimizes the cost (see Figure 3). However, this is not possible for larger cliques as they may in general share some cells of  $S_A$  and thus we defer them to the second phase.

Now observe that the most important vertices for computing the cost are the vertices constituting the set  $A$ . To see this just note that all other vertices see only  $A$  and their neighborhood (a clique) which is at least as large as for the vertices in  $A$ . It follows that we should only care about the number of selected vertices such that  $A$  is their cover set. Thus if the size of all cliques in  $\mathcal{C}$  is bounded in terms of  $k$  we are done. Alas, this is not the case.

We split the set  $\mathcal{C}$  into sets  $\mathcal{C}_1, \dots, \mathcal{C}_{2r}$ , and  $\mathcal{C}_{\max}$ . A clique  $C \in \mathcal{C}$  belongs to  $\mathcal{C}_{|C|}$  if  $1 \leq |C| \leq 2r$  and belongs to  $\mathcal{C}_{\max}$  otherwise. Note that cliques from  $\mathcal{C}_{\max}$  may be assigned only to cells having at least one index  $r+1$ . As mentioned we are about to design an ILP with a non-linear objective function. This ILP has variables  $x_{i,j}^q$  that express the number of cliques from the set  $\mathcal{C}_q$  assigned to the cell  $(i, j)$  of  $S_A$  (that is  $1 \leq i, j \leq r+1$  and  $q \in Q = \{1, \dots, 2r\} \cup \{\max\}$ ). The obvious conditions are the following (the  $\supseteq$  symbol translates to  $=$  if  $S(i, j) \leq \alpha$  while it translates to  $\geq$  if  $S(i, j) = \alpha + 1$ ).

$$\begin{aligned} \sum_{q \in Q} x_{i,j}^q &\supseteq S_A(i, j) && 0 \leq i, j \leq r+1 \\ \sum_{0 \leq i, j \leq r+1} x_{i,j}^q &= |\mathcal{C}_q| && \forall q \in Q \\ x_{i,j}^q &\geq 0 && 0 \leq i, j \leq r+1, \forall q \in Q \end{aligned}$$

We are about to minimize the following objective

$$\sum_{0 \leq i \leq r+1, 0 \leq j \leq r} \sum_{1 \leq q \leq 2r} (q-j)x_{i,j}^q + \sum_{0 \leq i \leq r+1} \sum_{\forall q} i \cdot x_{i,r+1}^q + g(x_{r+1,0}^{\max}, \dots, x_{r+1,r}^{\max}),$$

where  $g: \mathbb{N}^r \rightarrow \mathbb{N}$  is a function that has access to sizes of all cliques in  $\mathcal{C}_{\max}$  and computes the minimum possible assignment. We claim that the function  $g$  is a separable convex function in variables  $x_{r+1,0}^{\max}, \dots, x_{r+1,r}^{\max}$ . The first summand of the objective function describes the cliques of size at most  $2r$ . Their price corresponds to the number of vertices in the clique  $q$  minus the number of vertices that are not selected  $j$ . The second summand corresponds to the last row, where the cheapest price is always the number of selected vertices  $i$ . The last summand, discussed in the following paragraph, describes the assignment to the last row. The result then follows from Theorem 11 as the number of integral variables is  $O(r^3)$ .

Observe that the value of  $g(x_{r+1,0}^{\max}, \dots, x_{r+1,r}^{\max})$  is equal to sum of sizes of cliques “assigned to the last row” minus  $\sum_{j=0}^r j \cdot x_{r+1,j}^{\max}$ . Now,  $g(x_{r+1,0}^{\max}, \dots, x_{r+1,r}^{\max}) = g'(\sum_{j=0}^r x_{r+1,j}^{\max}) -$

$\sum_{j=0}^r j \cdot x_{r+1,j}^{\max}$ . Since all cliques in  $\mathcal{C}_{\max}$  are eligible candidates to be assigned to the last row and since it is always cheaper to assign there those of the smallest size among them we can define  $g'$  based only on the number of cliques that are assigned to the last row. This finishes the proof since  $g'$  is a convex function. See Section 2.2.1 for details on polynomial space version. ◀

Now we are ready to prove the main result of this section. It essentially follows by the exhaustive search among all possible shapes  $S$  such that  $\varphi$  is true under  $S$  and application of Lemma 12.

► **Proposition 13.** *Let  $G = (V, E)$  be a graph with twin cover  $K$  of size  $k$ . For an  $\text{MSO}_1$  formula  $\varphi(X)$  with one free variable that represents the set to be deleted it is possible to find a set  $W \subseteq V$  such that*

- $\varphi(W)$  holds in  $G$  and
- the cost of  $W$  is minimized among all subset of  $V$  satisfying  $\varphi(X)$  in time  $f(k, |\varphi|)|V|^{O(1)}$  for some computable function  $f$ .

**Proof.** We proceed as follows. For every possible selection of  $K \cap W$  we generate all possible shapes and check whether  $\varphi(X)$  evaluates to true under shape  $S$  and if so, we compute  $W$  for  $S$  having the minimal fair-cost. Finally, we return the set  $X$  minimizing the cost.

We stress that Lemma 8 applies only in cases where the cover set of the cliques at hand is not empty. Thus, these cliques are exceptional and we have to handle them separately. Note that in this case, the objective is different – we want to minimize the maximum number of selected vertices in any clique. It is not hard, however, to do this via standard tricks.

In order to summarize the running time we have

- at most  $2^k$  possible selections for  $W \cap K$  and
- at most  $2^k \cdot (\alpha + 1)^{(r+1)}$  possible shapes (for each such selection).

For every shape  $S$  we label the graph according to  $S$ . Afterwards, apply Lemma 8 and proposition 7 exhaustively on the labeled graph to resolve whether  $\varphi(X)$  evaluates to true under  $S$ . If  $\varphi(X)$  was evaluated to true, then using Lemma 12 on every possible  $A$  we obtain a set  $W_A$  minimizing the cost for vertices in  $A$  and put  $W = \cup_{A \subseteq K} W_A$ . Observe that this union gives the optimal cost for the selected shape  $S$ . Finally, we return the set  $W$  minimizing the cost for any true evaluated shape. Clearly this routine runs in FPT time with respect to  $k$  and  $|\varphi|$  as parameters. ◀

### 2.2.1 Polynomial Space Version of Proposition 13

We now argue that it is possible to implement Lemma 12 in polynomial space via reducing it to integer linear programming. We note that similar application of this technique is recently presented by Bredereck et al. [3].

► **Theorem 14** (Lenstra & Frank, Tardós [31, 12]). *There is an algorithm that given an ILP with  $p$  variables finds an optimal solution to it using  $O(p^{2.5p} \text{poly}(L))$  arithmetic operations and space polynomial in  $L$ , where  $L$  is the bitsize of the ILP.*

► **Remark 15.** Let  $G = (V, E)$  be a graph,  $K$  be a twin cover of  $G$ , and  $A \subseteq K$ . There is an algorithm that given an  $A$ -shape  $S_A$  of size  $r$  computes a valid solution for  $S_A$  of minimal cost in time  $f(|K|, r) \cdot |G|^{O(1)}$  and space polynomial in  $|G|$  or reports that  $S_A$  is not valid.

**Proof.** Here the only difference is that we have to rewrite the function  $g(x_{r+1,1}^{\max}, \dots, x_{r+1,r}^{\max})$  using a new variable  $y$  (representing its value) into constraints of ILP and thus obtain a linear objective.

We do this by adding a variable  $y \geq 0$  with constraint  $y = \sum_{j=1}^r x_{r+1,j}^{\max}$  and work with univariate function  $g(y)$  instead. We order cliques in  $\mathcal{C}_{\max}$  according to their size, that is,  $|C_1| \leq |C_2| \leq \dots \leq |C_t|$ , where  $t$  denotes the size of  $\mathcal{C}_{\max}$ . By  $c_i$  we denote the sum  $\sum_{j=1}^i |C_j|$  and note that  $g(y) = c_y$ . Finally we introduce a variable  $g_y$  representing the value of  $g(y)$  and add constraints

$$g_y \geq (y - i)c_i \quad \forall 1 \leq i \leq t.$$

Our result then follows as  $t \leq |V|$  and thus we add at most  $|V|$  new constraints.  $\blacktriangleleft$

### 3 The Fair VC problem

#### 3.1 Hardness for Treedepth and Feedback Vertex Set

We begin with several simple observations about the fair objective value  $k$  in the FAIR VC problem.

$\triangleright$  **Observation 16.** Let  $G = (V, E)$  be a graph and  $U \subseteq V$  be a vertex cover of  $G$  with fair objective  $k$ , that is,  $\forall v \in V$  it holds that  $|N(v) \cap U| \leq k$ . If  $v \in V$  has  $\deg(v) \geq k$ , then  $v \in U$ .

Note that we can use Observation 16 to enforce a vertex  $v$  to be a part of the fair vertex cover by attaching  $k + 1$  degree 1 vertices to  $v$ . Observe further that we may adjust (lower) the global budget  $k$  for individual vertex  $v$  by attaching vertices to  $v$  and then attaching  $k$  new leaves to the newly added vertices. To this end if the above operations are applied to a graph  $G$  resulting in a graph  $G'$ , then  $\text{td}(G') \leq \text{td}(G) + 2$  and  $\text{fvs}(G') = \text{fvs}(G)$ .

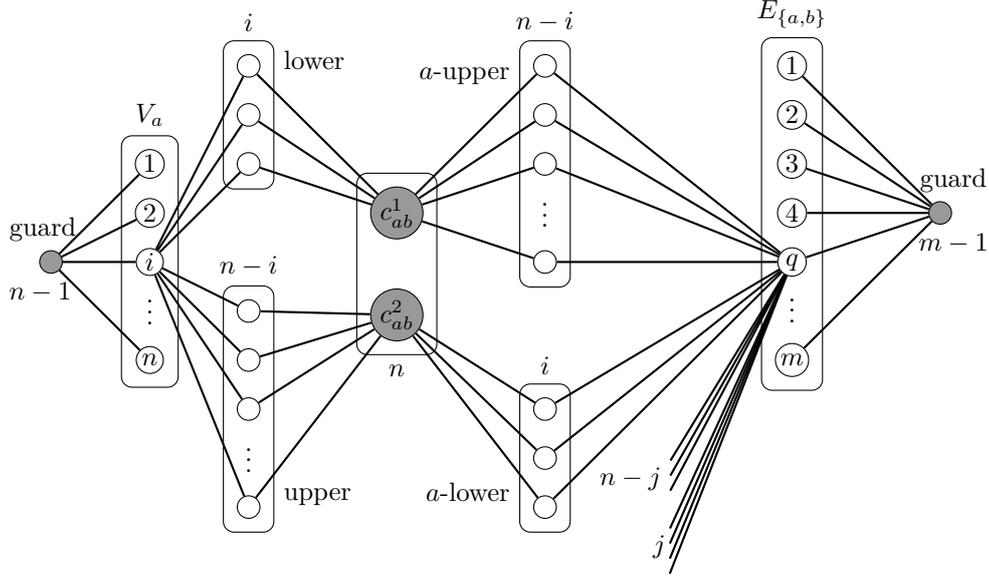
We observe substantial connection between FAIR VC and TARGET SET SELECTION (TSS). It is worth mentioning that VERTEX COVER can be formulated in the language of TSS by setting the threshold to  $\deg(v)$  for every vertex  $v$ . As a result, our reduction given here is, in certain sense, dual to the one given by Chopin et al. [5] for the TSS problem. However, we will show that the structure of the solution for FAIR VC is, in fact, the complement of the structure of the solution for TSS given therein. The archetypal  $W[1]$ -hard problem is the  $\ell$ -MULTICOLORED CLIQUE problem [9]:

|                             |   |
|-----------------------------|---|
| $\ell$ -MULTICOLORED CLIQUE | <i>Parameter: <math>\ell</math></i>   |
| <b>Input:</b>               | An $\ell$ -partite graph $G = (V_1 \cup \dots \cup V_\ell, E)$ , where $V_c$ is an independent set for every $c \in [\ell]$ and they are pairwise disjoint. |
| <b>Question:</b>            | Is there a clique of the size $\ell$ in $G$ ?   |

**Proof of Theorem 2.** Let  $G = (V_1 \cup \dots \cup V_\ell, E)$  be an instance of the  $\ell$ -MULTICOLORED CLIQUE problem and let  $n = |V_i|$  for all  $i \in [\ell]$ . We denote by  $E_{\{a,b\}}$  the set of edges between  $V_a$  and  $V_b$  and by  $m = |E_{\{a,b\}}|$ . We will describe graph  $H = (W, F)$  that together with  $k = \max(m - 1, 2n)$  will form an equivalent instance of the FAIR VC problem. The reduction has the following properties:

- $|W| = \text{poly}(n, k)$  and  $|F| = \text{poly}(m, k)$ ,
- $\text{td}(H) = \mathcal{O}(\ell^2)$ , and  $\text{fvs}(H) = \mathcal{O}(\ell^2)$ .

For an overview of the reduction please refer to Figure 4. There are three types of gadgets in our reduction, namely the vertex selection gadget (one for each vertex), the edge selection gadget (one for each edge), and the incidence check gadget (one for each vertex–edge incidence). We start by enumerating the vertices in each color class by numbers from  $[n]$  and edges by numbers in  $[m]$ . Throughout the proof  $a, b$  are distinct numbers from  $[\ell]$ .



■ **Figure 4** An overview of the reduction in the proof of Theorem 2. The gray vertices are enforced to be a part the fair vertex cover. If a vertex fair objective was lowered, then the resulting threshold is beneath the vertex (the group of vertices).

The  $V_a$  selection gadget consists of  $n$  *choice* vertices (representing the color class  $V_a$ ), a special vertex called *guard*, and a group of  $n^2$  *enumeration* vertices. The guard vertex is connected to all choice vertices, it is enforced to be a part of the fair vertex cover, and its budget is lowered so that at most  $n - 1$  choice vertices can be in any fair vertex cover. The  $i$ -th choice vertex is connected to  $n$  enumeration vertices. For each choice vertex there are  $n$  such vertices and so these are private for vertex  $i$ . We further divide these vertices into two parts – the *lower part* consists of  $i$  vertices and the *upper part* consists of  $n - i$  vertices.

The  $E_{\{a,b\}}$  selection gadget consists of  $m$  *choice* vertices, a special vertex called *guard*, and a group of  $2nm$  *enumeration* vertices and is constructed analogously to the vertex selection gadget. If the  $q$ -th edge of  $E_{\{a,b\}}$  connects  $i$ -th vertex in  $V_a$  and  $j$ -th vertex in  $V_b$ , there are (private)  $2n$  enumeration vertices are connected to the  $q$ -th choice vertex. These are split into four groups – *lower a-part* consisting of  $i$  vertices, *upper a-part* consisting of  $n - i$  vertices, and similarly *lower* and *upper b-parts*.

The  $ab$ -incidence check gadget consist of two vertices  $c_{ab}^1$  and  $c_{ab}^2$ . Both  $c_{ab}^1$  and  $c_{ab}^2$  are enforced to be a part of the solution and with a lowered budget in a way that at most  $n$  vertices in the neighborhood of each of them can be part of any fair vertex cover. The vertex  $c_{ab}^1$  is connected to every lower part vertex in the selection gadget for  $V_a$  and to every upper  $a$ -part vertex in the selection gadget for  $E_{\{a,b\}}$ . The vertex  $c_{ab}^2$  is connected to every upper part vertex in the selection gadget for  $V_a$  and to every lower  $a$ -part vertex in the selection gadget for  $E_{\{a,b\}}$ .

This finishes the construction of  $H$ . Now observe that if we remove vertices  $c_{ab}^1, c_{ab}^2$  from  $H$ , then each component of the resulting graph is a tree (rooted in its guard vertex) of depth at most 3. It follows that  $\text{td}(H) = \mathcal{O}(\ell^2)$  and  $\text{fvs}(G) = \mathcal{O}(\ell^2)$ , as so is the size of the removed set of vertices. We finish the proof by showing that the two instances are equivalent.

Suppose  $(G, \ell)$  is a yes-instance which is witnessed by a set  $K \subseteq V_1 \times \cdots \times V_\ell$ . We now construct a vertex cover  $C_K$  of  $H$  having  $|N(w) \cap C_K| \leq k$  for all  $w \in W$ . The set  $C_K$

contains the following:

- all enforced vertices (including all guard and check vertices),
- if  $v \in V_a \cap K$  is the  $i$ -th vertex of  $V_a$ , then all selection vertices of  $V_a$  but the vertex  $i$  are in  $C_K$  and lower and upper enumeration vertices of  $i$  are in  $C_K$ ,
- if  $v \in V_a \cap K$  and  $u \in V_b \cap K$  are adjacent through  $q$ -th edge of  $E_{\{a,b\}}$ , then all selection vertices of  $E_{\{a,b\}}$  but the vertex  $q$  are in  $C_K$  and  $q$ 's enumeration vertices are in  $C_K$ .

For the other direction we prove that a vertex cover  $C$  in  $H$  fulfils  $|N(w) \cap C| \leq k$  for all  $w \in W$  if it corresponds to a clique in  $G$ . For the other direction suppose that there is a vertex cover  $C$  in  $H$  such that  $|N(w) \cap C| \leq k$  for all  $w \in W$ . Recall that  $C$  has to contain all enforced vertices. This implies that at least 1 choice vertex for  $V_a$  is not in  $U$ ; we will show that exactly 1 such choice vertex is in  $U$ . The same holds for the edge choice vertices. To see this suppose for contradiction that 2 choice vertices for  $V_a$  (vertex  $i$  and  $j$ ) are not in  $U$ . Because  $U$  is a vertex cover of  $H$  it follows that their enumeration vertices must belong to  $U$ . But now take vertices  $c_{ab}^1, c_{ab}^2$ . In their neighborhood  $U$  have at least  $3n$  vertices ( $2n$  from the  $V_a$ 's numeration part and  $n$  from the  $a$ -numeration part of  $E_{\{a,b\}}$ ). This is absurd as these vertices (due to the lowered budget) can have at most  $2n$  vertices in their neighborhood and thus at least one of them exceeds its budget. Thus the selection gadgets actually encode some selection of vertices  $v_a$  and edges  $e_{a,b}$ . To finish the proof we have to observe that both  $c_{ab}^1$  and  $c_{ab}^2$  have at most (in fact, exactly)  $n$  neighbors in  $U$  if and only if the vertex  $v_a$  is incident to the edge  $e_{a,b}$ . If this holds for all possible combinations of  $a, b$ , then we have selected a clique in the graph  $G$ .

It remains to discuss the ETH based lower-bound. This follows straightforwardly from our reduction and the result of Chen et al. [4] who proved that there is no  $f(k)n^{o(\ell)}$  algorithm for  $\ell$ -MULTICOLORED CLIQUE unless ETH fails. Since we have  $\text{td}(G) + \text{fvs}(G) = O(\ell^2)$  in our reduction, the lower-bound follows. ◀

### 3.2 FPT algorithm for Modular Width

Since the algebraic expression  $A$  of width  $\text{mw}(G)$  can be computed in linear time [41], we can assume that we have  $A$  on the input. We construct the rooted ordered tree  $\mathcal{T}$  corresponding to  $A$ . Each node  $t \in \mathcal{T}$  is assigned a graph  $G^t \subseteq G$ , that is, the graph constructed by the subexpression of  $A$  rooted at  $t$ . Suppose we are performing substitution operation at node  $t$  with respect to template graph  $T$  and graphs  $G_1, \dots, G_r$ . Denote the resulting graph  $G^t$  and denote by  $n_i$  the size of  $V(G_i)$ .

**Proof of Theorem 3.** The edges in  $G^t$  between two vertices of  $G_i$  will be referred to as *old edges*, the edges between  $G_i$  and  $G_j$  for  $i \neq j$  (i.e., edges introduced by the template operation) will be referred to as *new edges*.

The computation will be carried out recursively from the bottom of the tree  $\mathcal{T}$ .

We first describe the structure of all vertex covers  $C$  in  $G^t$ . Observe that if  $ij \in E(T)$  then at least one of  $V(G_i), V(G_j)$  must be a subset of  $C$ ; otherwise there would be a new edge not covered by  $C$ . From this we can see that the set  $C_T := \{i : V(G_i) \subseteq C\}$  is a vertex cover of the template graph  $T$ . We call the  $C_T$  the *type of the vertex cover*  $C$ . Furthermore, every set  $C \cap V(G_i)$  must be a vertex cover of  $G_i$  (otherwise there would be an old edge uncovered by  $C$ ).

We now describe the fair cost of the cover  $C$  in terms of fair costs and sizes of the sets  $C \cap V(G_i)$ . Denote by  $c_i$  the size  $|C \cap V(G_i)|$  and by  $f_i$  the fair cost of  $C \cap V(G_i)$  in  $G_i$ . The *fair cost of  $C$  in  $W \subseteq V(G)$*  is defined as  $\max_{v \in W} |C \cap N(v)|$ . For  $i \in [r]$  the fair cost

of  $C$  in  $V(G_i)$  can be written as  $f_i + \sum_{j:ij \in E(T)} c_j$ . Clearly, fair cost of  $C$  is the maximum of the last expression over  $i \in [r]$ .

If we know the type  $C_T$  of the cover  $C$  this can be simplified based on whether  $i$  lies in  $C_T$ . If  $i \in C_T$  then  $f_i$  is  $\Delta(G_i)$  (the maximal degree of  $G_i$ ). If on the other hand  $i \notin C_T$  then all its neighbors  $j$  are in  $C_T$  and in this case  $c_j = n_j$ . Combining those observations, we have

$$\text{fair cost of } C \text{ in } G_i = \begin{cases} \Delta(G_i) + \sum_{j \notin C_T: ij \in E(T)} c_j + \sum_{j \in C_T: ij \in E(T)} n_j & i \in C_T, \\ f_i + \sum_{j: ij \in E(T)} n_j & i \notin C_T. \end{cases}$$

For each node  $t$  of the tree  $\mathcal{T}$  we keep an  $|V(G^t)|$  table  $\text{Tab}^t$  of integer values from  $[n] \cup \infty$ . The value at position  $\text{Tab}^t[p]$  is the smallest size of a cover in  $G^t$  of fair cost  $p$  or  $\infty$  if such cover do not exists.

The computation of  $\text{Tab}$  in leaves of  $\mathcal{T}$  is trivial. We describe how to compute value  $\text{Tab}^t[p]$  given that we know  $\text{Tab}_i$  in all children of  $t$ . It is enough to determine whether there exists a vertex cover  $C$  of  $G^t$  of fixed type; we can simply iterate over all types as there are at most  $2^r$  of them.

Fix a type  $C_T$ . The cover  $C$  of type  $C_T$  and fair cost at most  $p$  and  $\text{Tab}^t[p] \neq \infty$  exists if and only if for every  $i \notin C_T$  there is a vertex cover  $C_i$  of  $G_i$  of fair cost  $p_i$  such that  $\text{Tab}_i[p_i] \neq \infty$ . Moreover, we require that the values  $p_i$  satisfy the following inequalities:

$$p \geq \Delta(G_i) + \sum_{j \notin C_T: ij \in E(T)} \text{Tab}_j[p_i] + \sum_{j \in C_T: ij \in E(T)} n_j \quad \forall i \in C_T, \quad (1)$$

$$p \geq p_i + \sum_{j: ij \in E(T)} n_j \quad \forall i \notin C_T, \quad (2)$$

$$\text{Tab}^t[p] \geq \sum_{j \notin C_T} \text{Tab}_j[p_i] + \sum_{j \in C_T} n_j. \quad (3)$$

First, for every  $i \notin C_T$  we set  $p_i$  to the highest possible value without violating the inequality (2), that is  $p_i := p - \sum_{j: ij \in E(T)} n_j$ . Note that this is always a safe choice;  $p_i$  does not appear anywhere else in the constraints and choosing the highest possible value to give us greatest freedom due to the monotonicity of the table. Clearly, if any such  $p_i$  is negative we know that given constraints cannot be satisfied and there is no vertex cover  $C$  of given type and fair cost.

If for any  $i$   $\text{Tab}_i[p_i] = \infty$ , then there is no vertex cover  $C_i$  in  $G_i$  of fair cost  $p_i$ . This means that we cannot find cover  $C$  of given type and fair cost so we set  $\text{Tab}_i[p_i] = \infty$ . We check whether inequality (1) holds. If not, set  $\text{Tab}_i[p_i] = \infty$ . Otherwise we set  $\text{Tab}^t[p]$  be equal to the expression in (3) on the right side. We claim that there is a vertex cover  $C$  of given type and fair cost; we can set  $C = \bigcup_{i \in C_T} V(G_i) \cup \bigcup_{i \notin C_T} C_i$ , where  $C_i$  is any vertex cover of  $G_i$  of fair cost  $p_i$  and size  $\text{Tab}_i[p_i]$  (this is guaranteed to exist because  $\text{Tab}_i[p_i]$  was not  $\infty$ ). It is straightforward to check that  $C$  has required properties. Moreover, from our choice of values  $p_i$  it follows that a vertex cover of type  $C_T$ , fair cost  $p$  and size  $\text{Tab}^t[p]$  exists if and only if the described procedure finds values of  $p_i$ . By iterating over all types  $C_T$  we can fill the value  $\text{Tab}[p]$  as required.

To complete the description of the algorithm, it is enough to look whether there is not  $\infty$  value in  $\text{Tab}_{\text{root}}[k]$ , where  $k$  was the desired fair cost.

The running time is  $n$  for the induction over expression  $A$  times  $2^r$  different type of covers in any single node times filling the table of size at most  $n$  times  $nr$  for determining the correct values  $p_i$  and checking other inequalities for every  $i \in [r]$ . This altogether yields a  $2^r r n^3$  time algorithm.  $\blacktriangleleft$

**Proof of Lemma 4.** First we observe that modular width is trivially compositional, that is, for any two graphs  $G_1, G_2$  it holds that  $\text{mw}(G_1 \dot{\cup} G_2) = \max(\text{mw}(G_1), \text{mw}(G_2))$ , where  $\dot{\cup}$  denotes the disjoint union. Indeed this follows from the fact that disjoint union is one of the operations not affecting modular width. Now, it remains to show that FAIR VC is AND-compositional, see [9, Chapter 15]; the rest then follows from the framework of Bodlaender et al. [2]. To this end, observe that if a graph  $G$  is not connected, then  $U$  is a vertex cover in  $G$  if and only if  $U \cap C$  is a vertex cover in  $G[C]$  for every connected component  $C$  of  $G$ . ◀

#### 4 Hardness of Possible Extensions

We use the UNARY  $\ell$ -BIN PACKING problem as the starting point of our hardness reduction. UNARY  $\ell$ -BIN PACKING is W[1]-hard for parameter  $\ell$  the number of bins to be used [23]. Here, the item sizes are encoded in unary and the task is to assign  $n$  items to  $\ell$  bins such that the sum of sizes of items assigned to any bin does not exceed its capacity  $B$ . Formally, UNARY  $\ell$ -BIN PACKING is defined as follows.

|                           |   |
|---------------------------|---|
| UNARY $\ell$ -BIN PACKING | <i>Parameter: <math>\ell</math></i>   |
| <b>Input:</b>             | Positive integers $\ell$ and $B$ and item sizes $s_1, \dots, s_n$ encoded in unary. |
| <b>Question:</b>          | Is there a packing of all items into at most $\ell$ bins of size $B$ ?              |

**Proof of Theorem 5.** We construct a formula  $\varphi(X_1, \dots, X_\ell)$  and an instance  $(G, k)$  of FAIR VERTEX MSO EVALUATION with  $k = B$  from an instance of UNARY  $\ell$ -BIN PACKING as follows. The graph  $G$  is formed by  $n$  disjoint cliques and a universal vertex  $u$ . Cliques in  $G$  represent the items by their respective sizes, that is, there is a clique with  $s_i$  vertices for every  $i \in [n]$ ; denote the clique representing item  $i$  by  $C_i$ . This finishes the description of the graph; now we turn our attention to the formula. Free variables  $X_1, \dots, X_\ell$  are going to represent an assignment of items to bins. Note that it is possible to recognize the universal vertex  $u$  by the following FO formula, since  $u$  is the only vertex of  $G$  satisfying it:

$$\text{univ}(v) := (\forall w \in V)((w \neq v) \rightarrow (vw \in E)).$$

We fix  $u$  for the rest of the description of  $\varphi(X_1, \dots, X_\ell)$ ; this can easily be done by attaching  $(\exists u \in V)(\text{univ}(u))$  to it. Let  $p(v)$  be a predicate. For  $Q \in \{\exists, \forall\}$  we use the following  $Qv \in (V \setminus \{u\})(\psi(v))$  as a short form of the expression

$$(Qv \in V)((v \neq u) \rightarrow \psi(v)).$$

Note that this can be straightforwardly extended for more quantifiers.

In order to represent the bin choice, we need to ensure two conditions. First, every item is packed, that is, every non-universal vertex must belong to some  $X_j$ . Second, every item is fully packed inside (at least) one bin, that is, vertices belonging to the same clique agree on  $X_j$  membership. To do so we first introduce the following predicates (representing these conditions):

$$\text{cover}(X_1, \dots, X_\ell) := (\forall v \in V \setminus \{u\}) \left( \bigvee_{j=1}^{\ell} v \in X_j \right)$$

and

$$\text{same}(X_1, \dots, X_\ell) := (\forall v, w \in V \setminus \{u\}) \left( (vw \in E) \rightarrow \bigwedge_{j=1}^{\ell} (v \in X_j \Leftrightarrow w \in X_j) \right).$$

The construction of the new instance is finished by letting

$$\varphi(X_1, \dots, X_\ell) = (\exists u \in V)(\text{univ}(u)) \wedge \text{cover}(X_1, \dots, X_\ell) \wedge \text{same}(X_1, \dots, X_\ell).$$

It remains to argue that the instances are indeed equivalent. Note that the bin capacity/fair cost is essentially checked only for  $u$  since the fair cost of any vertex cannot exceed its degree. The degree of any other vertex (not  $u$ ) does not exceed  $\max_{i \in [n]} s_i$  and this is always upper-bounded by  $B$ .

Let  $(B, \ell, s_1, \dots, s_n)$  be a Yes-instance of  $\ell$ -UNARY BIN PACKING. This is witnessed by a mapping  $\sigma: [n] \rightarrow [\ell]$  assigning items to bins. Now, we put

$$W_j = \cup_{i \in [n]: \sigma(i)=j} C_i.$$

Observe that  $|N(u) \cap W_j| = \sum_{i \in [n], \sigma(i)=j} s_i \leq B$  for every  $j \in [\ell]$ , since  $\sigma$  represented a valid assignment. Furthermore,  $\varphi(W_1, \dots, W_\ell)$  holds, since every clique vertex is covered and vertices of a clique are always in the same  $W_j$ .

For the opposite direction assume we have graph  $G = (V, E)$  and that there exist sets  $W_1, \dots, W_\ell \subseteq V$  such that  $\varphi(W_1, \dots, W_\ell)$  holds in  $G$ . Recall that  $u$  is the universal vertex of  $G$ . First, we have that  $V = \{u\} \cup W_1 \cup \dots \cup W_\ell$ , since the predicate  $\text{cover}(W_1, \dots, W_\ell)$  holds if and only if every vertex in a clique  $C_i$  is in some  $W_j$ . Furthermore, since the predicate  $\text{same}(W_1, \dots, W_\ell)$  holds, we have that  $C_i \cap W_j$  is either an empty set or  $C_i$  for every  $i \in [n]$  and  $j \in [\ell]$ . This allows us to construct an assignment  $\sigma: [n] \rightarrow [\ell]$  by setting  $\sigma(i) = j$ , where  $j \in [\ell]$  is the smallest number such that  $W_j \cap C_i = C_i$  holds. Now, we have that  $\sum_{i \in [n], \sigma(i)=j} s_i \leq |W_j| \leq B$  by the fair objective. This finishes the proof since we have constructed a valid assignment.  $\blacktriangleleft$

**Proof of Theorem 6.** We construct a sentence  $\varphi$ , a graph  $G$ , and  $k$  forming an instance of FAIR EDGE FO DELETION with  $k = B$  from an instance of UNARY  $\ell$ -BIN PACKING as follows. Each item is represented by clique on  $3B$  vertices; denote the clique associated with  $i$ -th item by  $C_i$ . In addition, there are  $\ell$  vertices  $v_1, \dots, v_\ell$  (representing bins) and  $\ell$  guard vertices  $g_1, \dots, g_\ell$ . For each  $j \in [\ell]$  we connect  $v_j$  with exactly  $s_i$  vertices in  $C_i$  (call these vertices *special*) and with  $g_j$ . This finishes the description of  $G$ .

The sentence  $\varphi$  is constructed using auxiliary predicates which we describe first. A predicate  $\text{guard}(v)$  is used to recognize the guard vertices

$$\text{guard}(v) := (\exists u \in V)((uv \in E) \wedge (\forall w \in W \setminus \{u, v\})(vw \notin E)).$$

$$\text{bin}(v) := (\exists u)((uv \in E) \wedge \text{guard}(u)).$$

$$\text{item-edge} := (\forall v, w \in V : v \neq w)((\exists u \in V)(\neg \text{bin}(u) \wedge vu \in E \wedge wu \in E)) \rightarrow (vw \in E).$$

Recall that we set the fair cost to  $B$  and thus the solution  $F$  deletes at most  $B$  edges incident to any vertex. Note that a vertex  $v$  must be a guard vertex in order to fulfill the  $\text{guard}(v)$ , since the degree of every other vertex in  $G$  is at least  $3B$ . Suppose that the predicate  $\text{item-edge}$  holds in  $G \setminus F$ . We claim that  $vw \in E$  for any two vertices  $v, w \in C_i$ , since  $|C_i| = 3B$  it follows that  $v$  and  $w$  have at least  $B - 2$  common neighbors in  $C_i$ . Thus, the edge  $vw$  cannot be deleted, as  $v$  and  $w$  have a common (non-bin) neighbor (provided  $B \geq 3$ ). We define an auxiliary sentence

$$\psi = (\exists v_1, \dots, v_\ell \in V) \left( \bigwedge_{j \in [\ell]} \text{bin}(v_j) \wedge \bigwedge_{j \neq j' \in [\ell]} v_j \neq v_{j'} \right) \wedge \text{item-edge}$$

and, since  $\psi$  implicitly assures existence of  $\ell$  (different) guards, conclude the following claim.

▷ **Claim 17.** Let  $F$  be a set of edges in  $G$ . We have that  $G \setminus F \models \psi$  if and only if  $F$  contains only edges between bins and special vertices.

The next predicate we introduce is the  $\text{notable}(v)$ , defined as follows

$$\text{notable}(v) := \neg \text{guard}(v) \wedge ((\exists u \in V)(\text{bin}(u) \wedge uv \in E)).$$

▷ **Claim 18.**

1. If  $G \setminus F \models \text{notable}(v)$ , then  $v$  is a special vertex.
2.  $G \models \text{notable}(v)$  if and only if  $v$  is a special vertex.
3. If  $G \setminus F \not\models \text{notable}(v)$  for a special vertex  $v \in C_i$ , then  $N_{G \setminus F}(v) \subseteq C_i$ .

*Proof.* A vertex  $v$  is *notable* in a graph if  $\text{notable}(v)$  holds in that graph. The second part follows immediately from the construction of  $G$ , since the only vertices attached to the bin vertex  $v_j$  are the special vertices and its guard vertex  $g_j$  (which is not notable). On the other hand, the set  $F$  may contain edges  $vv_j$  for all  $j \in [\ell]$ . Clearly, such a former special vertex is not notable. Finally, if  $v$  is special and  $G \setminus F \not\models \text{notable}(v)$ , then  $v$  lost all of its edges to bin vertices  $v_1, \dots, v_\ell$ . ◁

The goal is to delete an edge set  $F$  that describes a valid assignment of items into bins. We need to ensure two conditions. First, every special vertex is not a neighbor of some bin vertex  $v_j$ . Second, if a special vertex  $v \in C_i$  is not a neighbor of a bin vertex  $v_j$ , then all special special vertices in  $C_i$  are not neighbors of  $v_i$ . The two conditions correspond to the following two predicates:

$$\text{cover} := (\forall v \in V)(\exists u \in V)(\text{bin}(u) \wedge uv \notin E)$$

$$\begin{aligned} \text{same} := & (\forall v, w \in V) \left( (\text{notable}(v) \wedge \text{notable}(w) \wedge vw \in E) \rightarrow \right. \\ & \left. \rightarrow (\forall u \in V : \text{bin}(u))(uw \in E \Leftrightarrow vw \in E) \right) \end{aligned}$$

Now, we are ready to give the sentence  $\varphi$  that describes the problem:

$$\varphi = \psi \wedge \text{same} \wedge \text{cover}.$$

This finishes the description of the reduction. We are left with validating that the two instances are equivalent.

Suppose we were given a Yes-instance of  $\ell$ -UNARY BIN PACKING and let  $\sigma: [n] \rightarrow [\ell]$  be the assignment of items to bins witnessing this fact. The set  $F$  contains an edge  $uv_j$  for a special vertex  $u \in C_i$  and a bin vertex  $v_j$  whenever  $\sigma(i) = j$ . Clearly, we have  $|\{e \in F : e \ni v_j\}| \leq \sum_{i \in [n], \sigma(i)=j} s_i \leq B = k$  for every vertex  $v_j$  with  $j \in [\ell]$ , while every other vertex has at most one incident edge in  $F$ . Now, we have to verify that  $G \setminus F \models \varphi$ . We have  $G \setminus F \models \text{same} \wedge \text{cover}$ , since  $\sigma$  is an assignment. Finally,  $G \setminus F \models \psi$ , since our  $F$  fulfills the condition of Claim 17.

Suppose now that we have a set  $F$  of edges of  $G$  with fair cost  $B$  such that  $G \setminus F \models \varphi$ . By Claim 17 we have that  $F$  contains only edges between special vertices in  $G$  and bin vertices  $v_1, \dots, v_\ell$ . We partition the set of special vertices into sets  $N$  and  $R$ ; we put a special vertex  $v$  in  $N$  if  $G \setminus F \models \text{notable}(v)$  and we put it in  $R$  otherwise. Note that  $|R| \leq B$ , since a vertex in  $R$  contributes to the fair cost of every bin vertex  $v_j$ . By Claim 18 and the fact that  $G \setminus F \models \text{same}$  we have that a bin vertex  $v_j$  is either completely attached or non-attached to  $C_i \cap N$  for every  $j \in [\ell]$  and every  $i \in [n]$ . Furthermore, there are no edges between a vertex

in  $R$  and a bin vertex  $v_j$  in  $G \setminus F$  for every  $j \in [\ell]$ . We define the assignment  $\sigma: [n] \rightarrow [\ell]$  by defining  $\sigma(i)$  to be the smallest integer such that there are no edges between  $v_{\sigma(i)}$  and  $C_i$  in  $G \setminus F$ . Since  $\sigma$  is an assignment by  $G \setminus F \models \text{cover}$ , it remains for verify that the capacity condition is fulfilled. For that we have

$$\sum_{i \in [n], \sigma(i)=j} s_i \leq |\{e \in F: v_j \in e\}| \leq \sum_{i \in [n], \sigma(i)=j} |N \cap C_i| + |R| \leq k = B.$$

We conclude that  $\sigma$  is a valid assignment and the theorem follows.  $\blacktriangleleft$

## 5 Conclusions

**Fair Edge L Deletion problems.** The crucial open problem is to resolve the parameterized complexity of the FAIR FO EDGE DELETION problems for parameterization by neighborhood diversity and twin cover. Observe that there is a big difference between vertex and edge deletion problems—in our hardness reduction we use a deletion to an edgeless graph but a fair edge cost, in this case, equals to the maximum degree of the former graph (and thus it is computable in polynomial time).

**Generalization of parameters.** Another open problem is to resolve the parameterized complexity of the FAIR MSO<sub>1</sub> VERTEX EVALUATION problems with respect to graph parameters generalizing neighborhood diversity or twin cover (e.g., modular width or cluster vertex deletion number respectively).

It seems that a more careful analysis of the model-checking algorithm may yield (again) sufficient insight into the structure of the fair solution and thus lead to an FPT algorithm for this wider class of graphs. Though, it remains open whether this is possible and there is an FPT algorithm for FAIR MSO VERTEX EVALUATION for this parameterization or not.

**MSO with Local Linear Constraints.** Previously, an FPT algorithm for evaluation of a fair objective was given for parameter neighborhood diversity [34]. That algorithm was extended [25] to a so-called *local linear constraints* again for a formula  $\varphi(\cdot)$  with one free variable that is defined as follows. Every vertex  $v \in V(G)$  is accompanied with two positive integers  $\ell(v), u(v)$ , the lower and the upper bound, and the task is to find a set  $X$  that not only  $G \models \varphi(X)$  but for each  $v \in V(G)$  it holds that  $\ell(v) \leq |N(v) \cap X| \leq u(v)$ . Note that this is a generalization as fair objective of value  $t$  can be tested in this framework by setting  $\ell(v) = 0$  and  $u(v) = t$  for every  $v \in V(G)$ . Is this extension possible for parameterization by the twin cover number?

To support this question we note that in the proof of Lemma 12 the minimal size of the neighborhood of  $B$  for a shape in exact neighborhood of  $B$  is computed. It is not hard to see that through a similar argument we can compute the maximal size of the neighborhood of  $B$  for a shape in the exact neighborhood of  $B$ . Furthermore, lower and upper bounds for vertices in a clique can be assumed to be nearly the same—each differs by at most 1 [25]. Thus, Lemma 12 gives only that if at least one of the computed bounds for a vertex  $v$  in the twin cover is within  $\ell(v)$  and  $u(v)$ , then there is a solution with desired properties. However, if on the other hand, it happens that both  $\ell(v), u(v)$  are in between the computed values, we do not know whether or not any of the desired values are attainable. To see that not all values in the thus computed range are attainable one can construct a formula that for twin cliques up to a certain size check that the number of selected vertices is even. Then, if the input graph contains only cliques up to this size no twin cover vertex has an odd number of neighbors in the set  $X$  (provided the cover vertices form an independent set).

**Towards new fair problems.** As we proposed the examination of FAIR VC already, we would like to turn an attention to exploring fair versions of other classical and well-studied VERTEX DELETION problems. In contrast, certain FAIR EDGE DELETION problems have got some attention before, namely FAIR FEEDBACK EDGE SET [33] or FAIR EDGE ODD CYCLE TRANSVERSAL [27]. Besides FAIR VC we propose a study of FAIR DOMINATING SET and FAIR FEEDBACK VERTEX SET. In particular, it would be really interesting to know whether fair variants of VERTEX COVER and DOMINATING SET admit a similar behavior as in the classical setting.

Furthemore, We would like to ask whether there is an NP-hard Fair Vertex Deletion problem that admits an FPT algorithm for parameterization by treedepth (and feedback vertex set) of the input graph.

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