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ABSTRACT OF DOCTORAL THESIS

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Variants of graph labeling problems

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Contents

1 Introduction

In this thesis, I present three selected results obtained during my Ph.D. studies. In addition, I attach a short annotated bibliography about fair problems, the topic of the third result. All the results are presented in the original journal form with only a small adjustment of their layout.

The first result appeared in Information Processing Letters journal: *Minki Kim, Bernard Lidický, Tomáš Masařík, and Florian Pfender. Notes on the complexity of packing coloring. Information Processing Letters, 137:6–10, 2018. doi:10.1016/j.ipl.2018.04.012.* [42]. The second was published on ISAAC 2018 conference and is currently under reviews in a journal: *Tereza Klimošová, Josef Malík, Tomáš Masařík, Jana Novotná, Daniël Paulusma, and Veronika Slívová. Colouring (P_r+P_s)-free graphs. In 29th International Symposium on Algorithms and Computation, ISAAC 2018, December 16–19, 2018, Jiaoxi, Yilan, Taiwan, pages 5:1–5:13, 2018. doi:10.4230/LIPIcs.ISAAC.2018.5.* [43]. The submitted journal version with full proofs is also available on arxiv [44]. The third has just been accepted to MFCS 2019 conference: *Dušan Knop, Tomáš Masařík, and Tomáš Toufar. Parameterized Complexity of Fair Vertex Evaluation Problems. In 44rd International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, August 26–30 2019, Aachen, Germany, pages 8:1–8:16, 2019.* [47]. We are about to submit a full version to the journal and this version is also available on arxiv [46].

My thesis is centered around graph labeling problems. Those are variants of the famous graph coloring problem that was first mentioned as early as in the 19th century. The original task is to assign colors to vertices of the graph such that the adjacent vertices obtain different colors. This problem is sometimes referred to as *proper coloring*. The graph coloring problem emerges in countless practical application varying from the scheduling to the register assignment. Moreover, it is an important theoretical tool, and therefore, many classical results are described in the language of coloring and its modifications. Graph labeling usually means that we exchange the set of indistinguishable colors for a set of natural numbers so that some additional special meaning can be tied to them.

1.1 Graph Problems and Decision Problems

In this work, we study problems on graphs. A graph is a basic structure consisting of the set of vertices and of the set of edges where each edge represents a pair of vertices. We refer to textbooks and monographs [17, 7, 6, 63] for the usual concepts and notation of graph theory used in this thesis.

A decision problem, or generally, an algorithmic problem, is a language over a finite alphabet. We give a similar formal definition for a parameterized problem (see Definition 1). In the case of graphs, the language is just a set of graphs that satisfy the problems, i.e., the set of graphs such that the answer to the decisional problem is yes. Loosely speaking, we are usually interested in problems that can be solved by computers in asymptotically reasonable time.

This vague idea leads to well-studied models of Turing machines and definitions of P and NP classes. This is a well-established theory, described in many books, e.g., in the monograph by Garey and Johnson [36], or the original papers by Cook [10] and Karp [41] from the '70s. We assume that the reader is familiar with these concepts, and refer to [36] for formal definitions.

1.2 Graph Colorings and Labelings

The area of graph coloring, labelings, and related research is one of the most studied among the whole graph theory. There are several books dedicated to graph coloring e.g., [40] and [55]. Also general surveys, for instance [31], as well as several specialized surveys, e.g., [38, 9, 62], were published.

For a graph G let L be a function $L : V(G) \to 2^{\{1,\dots,c\}}$, where L is usually called a *list assignment*, then graph G is *L-colorable* (or *list-colorable*) if there exists a proper coloring $\varphi : V(G) \to \{1, \dots, c\}$ such that $\varphi(v) \in L(v)$ for all vertices v. Note that if for all vertices the list is the same and contains all colors $\{1, \dots, c\}$, then this is just the proper coloring. We define a special variant called *list k-coloring* where for all vertices the lists are subsets of $\{1, \dots, k\}$. In this language, the *precoloring extension* problem allows for all vertices only list $\{1, \dots, k\}$ or just a singleton containing only one color. On the other hand, *k-list coloring* or *k-choosability* bound only the size of the lists (each list has size at least k) but not the number of allowed colors.

Besides a part of my work, I present in detail in the main part of the thesis, I was during my PhD studies involved in another project closely related to the graph coloring problem. The following subsection serves as a small excursion to the concept of flexibility in the graph coloring.

1.2.1 Flexibility

In certain applications of the classical coloring it is common that some vertices have preferred resource(s). This idea motivated precoloring extension questions. However, unfortunately, it is not usually possible to satisfy all such preferences. The notion of ε-Flexibility was first defined by Dvořák, Norin, and Postle in [26]. Instead of satisfying all the preferences, the aim is to satisfy at least a constant fraction of any request.

More formally, we are given a graph together with its list assignment. By *request* we mean a set of pairs, vertex and its preferred color. Note that not necessarily every vertex has to make a request. Graph G is *ε-flexible*, if it is possible for any list assignment L and any request, L-color the graph G such that at least a constant fraction of the request is satisfied. As it turns out, this question is trivial in the ordinary proper coloring setting with a bounded number of colors (where all the lists are of the same size and they consist of colors ranging from 1 to k). The answer is always positive there, since we can permute the colors according to the request, and therefore, satisfy at least $\frac{1}{k^2}$ fraction of any request. On the other hand, flexibility brought about a number of interesting problems in the list coloring setting. The main target we aimed for is the following statement. There exists an absolute constant ε such that for any graph in the studied graphs class with any lists assignment of size k, and for any request (preferred colors of some of the vertices) there exists a list coloring of the graph such that it satisfies at least ε fraction of the request. A well-studied notion of choosability forms a trivial lower-bound on the value of k from the previous statement. We subsequently derived a couple of theorems of the mentioned form on various subclasses of planar graphs [25, 24, 54]. Table 1 summarizes known results and provides a comparison with the choosability on planar graphs.

1.3 Hereditary Graph Classes

Hereditary property is one of the main themes in the study of mathematical structures. *Hereditary graphs* are graph classes such that are closed under vertex deletion. In particular,

Planar Graphs	General	Triangle-free	C_4-free	Girth 5	Girth 6
Choosability	5 [60]	4	4 [48]	3 [61]	3
List size in ε-Flexibility	6 [26]	4 [25]	5 [54]	4	3 [24]

Table 1: A summary and a comparisons of choosability and respective results in the flexibility setting on subclases of planar graphs. Non-implied bounds are accompagned with the respective citation.

pattern-free graphs are characterized by some forbidden pattern. This notion captures a large number of well-studied graph classes that are hereditary. For example, bipartite graphs, chordal graphs, and many others. It has a connection with minor-closed graph classes (for example, planar graphs) that can be viewed as even a stronger notion of excluding some patterns.

In particular, we consider *H-free graphs*, i.e., graphs without an induced copy of graph H. Those graphs are obviously hereditary. Despite the easy description, such graphs are quite difficult to analyze and it seems that a deeper understanding of these graph classes or new tools should be developed.

1.4 Parameterized Complexity

The central concept of the thesis is parameterized complexity, which is one of the main tools in the study of algorithms and complexity, nowadays.

As we discussed, many classical problems are NP-complete [36], and therefore, they are not well-scalable even for modern computers. In the run to overcome this problem, many approaches are considered. We can either sacrifice the optimality of the solution, and therefore, aim for approximation algorithms. Or we can find an additional structure in the input data and measure the time complexity in both, the size of the input and the chosen parameters. Those algorithms are called *Parameterized Algorithms*. The history of this field started by a series of paper by Downey and Fellows [18, 20, 21, 19, 1] with the very first paper that appeared on FOCS conference in 1989 [2]. Since the publication of a seminal book [22] by Downey and Fellows in 1999, Parameterized Complexity became one of the most important fields in algorithm study with plenty of essential publications each year and a handful of books, e.g., [23, 29] among others. The very recent development in the field is covered in a book by Cygan et al. [14].

First, we define a parameterized problem.

Definition 1. *A* parameterized problem *is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed alphabet and Σ^* is an arbitrary string over the alphabet. Then for an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, k is called the* parameter.

The class of the effective parameterized algorithms is called *Fixed Parameter Tractable*, in short *FPT*. It contains all the algorithms with running time $f(k)n^c$, where f is an arbitrary computable function, k represents the parameter, n is the size of the input, and c is a constant. Less effective is the class of XP algorithms with running time $n^{f(k)}$. However, for problems where an FPT algorithm probably does not exist (W-hard problems, see Subsection 1.4.1), XP algorithms might still be a very good choice.

To support the importance of the field in practical computations, we provide Table 2 that sums up the differences in the running time of the efficient algorithms—a comparison of both FPT and XP with a naive single exponential algorithm.

k/n	50 (1 day)	100 (3.2T yrs)	500 (10^{133} yrs)	1000 (10^{283} yrs)
5	133 ns | 26 ms	266 ns | 0.8 s	1.3 μs | 43 mins	2.7 μs | 1 day
10	4 μs | 94 days	8 μs| 264 yrs	40 μs | 2.5G yrs	80 μs | 2.6T yrs
25	0.1 s | 10^{24} yrs	0.2 s | 10^{32} yrs	1 s | 10^{49} yrs	2 s | 10^{57} yrs

Table 2: Comparisons of running times between naive exponential (2^n), XP (n^k) and FPT ($2^k n$) algorithms on a current computer (4 cores, 3GHz). Naive running times are in the first row in brackets, FPT | XP running times are within the inner-fields of the table. Shortcut yrs stands for years.

1.4.1 Parameterized Reduction

It is usually useful to derive hardness results alongside with the development of algorithms. On the first glance, it seems that whenever we have a parameter, such that the problem is NP-hard even when the parameter is constant, then the problem is hard for parameterization. More formally, if the problem is NP-hard for all values of a parameter larger than some constant, then it is called *para*-NP-*hard*. This reasoning rules out the existence of not only FPT algorithms but also XP algorithms (unless P = NP).

We define the parameterized reduction that is useful to show evidences that some problems would unlikely allow an FPT algorithm.

Definition 2. *Let A, B be two parameterized problems.* Parameterized reduction *is an algorithm that given an instance (x, k) of A outputs an instance (x', k') of B such that:*

- *(x, k) is yes-instance of A if and only if (x', k') is yes-instance of B,*
- *$k' \leq g(k)$, for a computable function g,*
- *the reduction algorithm runs in time $f(k) \cdot |x|^{\mathcal{O}(1)}$, for a computable function f.*

The parameterized reduction is similar to the classical polynomial-time reduction that is used to show NP-hardness. However, in fact, both reductions are incomparable, although many existing reductions fit both contexts. It can be observed that the parameterized reduction preserves containment in FPT class, i.e., if there is a parameterized reduction from A to B and problem B is in FPT then also problem A is in the FPT class.

Since even P $\neq$ NP is far from being proven then also a classification of problems hard for parameterized algorithms has to be based on some assumption. Usually, even a bit stronger assumption than P $\neq$ NP is used. There is a nested hierarchy of problems, where all the problems in one class are equivalent under a parameterized reduction:

$$\text{FPT} = \mathsf{W[0]} \subseteq \mathsf{W[1]} \ldots$$

Problems in classes $\mathsf{W[t]}$ for $t \geq 1$ are unlikely to admit an FPT algorithm. Details of the definition of W-hierarchy is omitted s for our purposes it suffices to refer to problems that have been shown $\mathsf{W[t]}$-hard elsewhere. For a more detailed introduction to the topic consult the book [14] or the original paper [20].

1.4.2 Kernelization

Kernelization plays an important role in many results from the parameterized algorithms field.

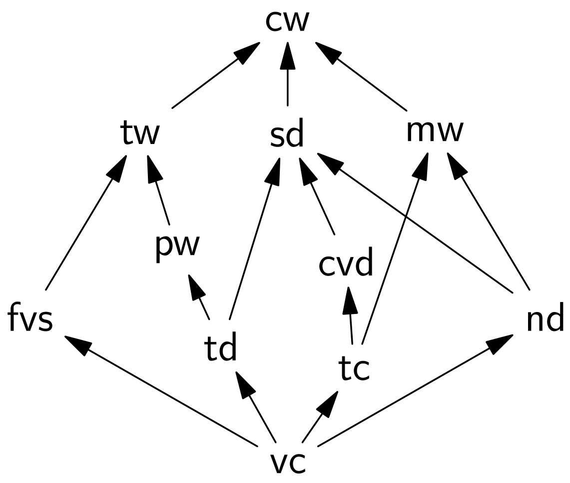

Figure 1: Hierarchy of graph parameters considered in the thesis. An arrow indicates that a graph parameter upper-bounds the other.

Definition 3. *Let L be parameterized problem. A* kernelization algorithm *or* kernelization *is an algorithm such that for any instance (x, k) it outputs in time polynomial in $|(x, k)|$ a string $x' \in \Sigma^*$ and an integer $k' \in \mathbb{N}$ such that*

$$(x, k) \in L \Leftrightarrow \Big((x', k') \in L \textit{ and } |x'|, k' \leq h(k)\Big)$$

where h is an arbitrary computable function. Function h is called the size *of the kernel.*

Definition 3 is sometimes strenghtened in a way that $k' \leq k$. It is well-known that the problem is in FPT if and only if it has a kernel [8]. One implication is easy since the kernelization algorithm takes polynomial time and then the problem can be solved exhaustively on the kernel in time independent on the original size of the input and dependent only on the parameter(s).

In particular, a big effort was spent in obtaining polynomial kernels or in negative result—refuting the existence of a polynomial kernel. Kernelization algorithm is described in the proof the main theorem in Section **??**. Refuting the polynomial kernel is briefly shown in Lemma **??**. Very recently (2019), the entire book devoted to Kernelization has appeared [30]. It is a good source of further details.

1.4.3 Structural Graph Parameters

In this work, we considered structural parameters in Chapters **??**, **??**, and **??**. See Figure 1 for an overview of the parameters.

We define structural parameters for a graph $G = (V, E)$. The most famous structural parameter is treewidth. The history of this parameter reaches back to Bertelè and Briochy in 1972 [4]. A modern equivalent defintion works with a term called *tree decomposition* of graph G. It is defined as a tree T with nodes $X_i \subseteq V(G)$, satisfying the following properties:

- $V(G) = \cup_i X_i$.
- For each $\{u, v\} \in E(G)$ exists $t \in V(T)$ such that $u, v \in X_t$.
- For each $v \in V(G)$, nodes that contain the vertex v induce a connected subgraph in T.

The *size* of a tree decomposition is the maximum size of the node. Then the *treewidth* ($\mathsf{tw}(G)$) is the minimum size of a tree decompostion taken over all tree decompositions of the graph G. Computing the exact value the treewidth is NP-complete [3], however, it can be computed in FPT time parameterized by its size [5]. We note for curiosity that the two years of an annual PACE challenge competition were devoted to computing the treewidth as good as possible [15, 16].

The *pathwidth* ($\mathsf{pw}(G)$) is the minimum size of the tree decompositions of graph G, where the decomposition is restricted only to a simple path. This gives a trivial inclusion. Even weaker graph parameter is the *treedepth* of a graph G ($\mathsf{td}(G)$). It is defined as the minimum height of a rooted forest whose transitive closure contains the graph G [56]. The *feedback vertex set* ($\mathsf{fvs}(G)$) is the minimum number of vertices of a graph G whose removal leaves a graph without cycles. The simplest considered parameter is *vertex cover* ($\mathsf{vc}(G)$), whih is the minimum number of vertices of a graph G, whose removal leaves an edgeless graph. All the above-mentioned parameters form so-called sparse parameters.

Dense graph parameters follow—as oposed to sparse, cliques has the following parameters bounded. The *neighborhood diversity* ($\mathsf{nd}(G)$) is the smallest integer r, such that the graph can be partitioned into r sets, where each set is either complete graph or an independent set and each pair of sets forms either a complete bipartite graph or there is no edge between them.

A more complicated conept is the *modular width* of a graph G ($\mathsf{mw}(G)$), whih is the smallest positive integer r such that G can be obtained from an algebraic expression of width at most r, defined as follows. The *width of an expression* A is the maximum number of operands used by any occurrence of the substitution operation in A, where A is an algebraic expression that uses the following operations:

1. Create an isolated vertex.

2. The *substitution operation* with respect to a template graph T with vertex set $[r]$ and graphs $G_1, \dots, G_r$ created by algebraic expression. The substitution operation, denoted by $T(G_1, \dots, G_r)$, results in the graph on vertex set $V = V_1 \cup \cdots \cup V_r$ and edge set $E = E_1 \cup \cdots \cup E_r \cup \bigcup_{\{i,j\} \in E(T)} \Big\{ \{u, v\} : u \in V_i, v \in V_j \Big\}$, where $G_i = (V_i, E_i)$ for all $i \in [r]$.

An algebraic expression of width $\mathsf{mw}(G)$ can be computed in linear time [59]. The *twin cover* $\mathsf{tc}(G)$ (introduced by Ganian [33]) is one possible generalization of vertex cover. It is defined as the number of vertices needed to cover all edges of graph G that are not twin-edges; an edge $\{u, v\}$ is a *twin-edge* if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. Here, we measure the number of vertices needed to cover all edges that are not twin-edges; an edge $\{u, v\}$ is a *twin-edge* if $N(u) \setminus \{v\} = N(v) \setminus \{u\}$. A more general parameter than twin cover is *cluster vertex deletion* ($\mathsf{cvd}(G)$), that is, the smallest number of vertices one has to delete from a graph in order to get a collection of (disjoint) cliques. The *shrub dept* ($\mathsf{sd}(G)$) is defined in [34] and since it is here only for completeness of the picture, we omit here the formal definition. The last presented and the most powerfull is the *cliquewidth* ($\mathsf{cw}(G)$) that upper-bounds all the studied parameters. This concept has been defined in [13] already in 1993. It is defined as the minimum number of colors used in the following process. Starting with an empty graph, the given graph G is created by the following operations.

- Create a vertex of a certain color.
- Make the disjoint union of two colored graphs.

- Add all the edges between vertices of two different colors.
- Change color of all the vertices of a certain color to a different color.

1.5 Graph Logics

Graph properties can be formally specified and modeled by a suitable logic. We provide here only a brief and rather an informal introduction to the subject. For more extensive description We recommend book [51] which explores the use of logic from graphs to finite models. In the basic setting, a logic has access to the graph and it can make queries, whether two vertices are adjacent or not. Standard and well-examined logics for graphs are MSO_2, MSO_1, and FO. The simplest among them is the FO logic. There the only allowed quantification is over elements (vertices or edges) of the graph. A strictly more powerful is MSO_1 logic, where in addition the quantification over the sets of vertices is possible. The most powerful out of them is MSO_2, where in addition even the quantification over the set of edges is allowed. For example, the hamiltonicity (the existence of a path visiting all the vertices of a given graph) can be expressed in MSO_2 but not in MSO_1 [12, 51]. The connectivity in the graph is an example of a property, that can be expressed in MSO_1 but not in FO (not even in existential MSO_2, where only existential quantification is allowed) [27].

1.5.1 Graph Metatheorems

Undoubtedly, Courcelle's Theorem [11] for graph properties expressible in the monadic second-order logic (MSO_2) on graphs of bounded treewidth, as well as an MSO_1 algorithm on graphs of bounded clique-width, play a prime role among model checking algorithms. In particular, Courcelle's Theorem provides for an MSO_2 sentence φ an algorithm, that given an n-vertex graph G with treewidth k decides, whether φ holds in G in time $f(k, |\varphi|)n$, where f is some computable function and $|\varphi|$ is the quantifier depth of φ. In other words, we obtain an FPT algorithm parameterized by treewidth and quantifier depth of the formula. Similar results are known for a weaker MSO_1 logic and clique-width. We cannot hope for much more on dense graph classes since MSO_2 model checking is not even in XP on graphs as simple as cliques unless E = NE [50].

There are many more FPT model checking specialized algorithms, e.g., an algorithm for (existential counting) modal logic model checking on graphs of bounded treewidth [57], MSO model checking on graphs of bounded neighborhood diversity [49], or MSO model checking on graphs of bounded shrubdepth [35] (generalizing the previous result). First order logic (FO) model checking received recently quite some attention as well and algorithms for graphs with bounded degree [58], nowhere dense graphs [39], and some dense graph classes [32] were given.

2 Our contribution

2.1 Complexity of Packing Coloring

A graph labeling problem that has its motivation in assigning frequencies to transmitters, under its original name, Broadcasting Chromatic Number. It was first formulated by Goddard, Hedetniemi, Hedetniemi, Harris, and Rall [37]. A packing k-coloring for some integer k of a graph $G = (V, E)$ is a mapping $\varphi : V \to \{1, \ldots, k\}$ such that any two vertices u, v of color $\varphi(u) = \varphi(v)$ are in distance at least $\varphi(v) + 1$. The packing chromatic number of G is the smallest k such that there exists a packing k-coloring of G.

Fiala and Golovach [28] showed that determining the packing chromatic number for chordal graphs is NP-complete for diameter exactly 5. While the problem is easy to solve for diameter 2, we show NP-completeness for any diameter at least 3. Our reduction also shows that the packing chromatic number is hard to approximate within $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$.

In addition, we design an FPT algorithm for interval graphs of bounded diameter. This leads us to exploring the problem of finding a partial coloring that maximizes the number of colored vertices.

2.2 Coloring of H-free Graphs

We examined H-free graphs, that are graphs without an induced copy of H. We showed that list 3-coloring is polynomial time solvable on $P_2 + P_5$-free and $P_3 + P_4$-free graphs where P_i represents a path on i vertices and symbol + denotes disjoint union. By this result, we completed the characterization for 3-coloring of H-free graphs for any H up to seven vertices.

2.3 Survey on Fair problems

Both Packing coloring and Coloring of H-free graphs are very heavily studied and several overview papers were published, see e.g., [38]. However, Fair problems have not received much attention yet, therefore we add a short annotated summary of known results from this field. This summary also contains connections with a closely related topic of defective coloring.

2.4 Methatheorems for Fair Problems

It is a part of a larger project on studying fair problems and their extensions. They have been mostly studied from the metatheorem point of view in papers [45, 53, 47].

A prototypical graph problem is centered around a graph-theoretic property for a set of vertices and a solution to it is a set of vertices for which the desired property holds. The task is to decide whether, in the given graph, there exists a solution of a certain quality, where we use size as a quality measure. In this work, we are changing the measure to the fair measure (cf. Lin and Sahni [52]). The fair measure of a set of vertices S is (at most) k if the number of neighbors in the set S of any vertex (in the input graph) does not exceed k. One possible way to study graph problems is by defining the property in a certain logic. For a given objective, an evaluation problem is to find a set (of vertices) that simultaneously minimizes the assumed measure and satisfies an appropriate formula.

More formally, we study the MSO Fair Vertex Evaluation, where the graph-theoretic property is described by an MSO formula.

In the presented paper we show that there is an FPT algorithm for the MSO Fair Vertex Evaluation problem for formulas with one free variable parameterized by the twin cover number of the input graph and the size of the formula. One may define an extended variant of MSO Fair Vertex Evaluation for formulas with ℓ free variables; here we measure a maximum number of neighbors in each of the ℓ sets. However, such variant is W[1]-hard for parameter ℓ even on graphs with twin cover one.

Furthermore, we study the Fair Vertex Cover (Fair VC) problem. Fair VC is among the simplest problems with respect to the demanded property (i.e., the rest forms an edgeless graph). On the negative side, Fair VC is W[1]-hard when parameterized by both treedepth and feedback vertex set of the input graph. On the positive side, we provide an FPT algorithm for the parameter modular width.

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3 List of my Publications

By convention of my research area, author names are in alphabetical order, except Papers [10] and [13].

3.1 Journal Publications

[1] Tomáš Masařík and Tomáš Toufar. Parameterized complexity of fair deletion problems. In *Discrete Applied Mathematics*, available online, 2019. doi:10.1016/j.dam.2019.06.001.
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3.2 Conference Publications

(excluding those published in journals)

[5] Tomáš Masařík, Irene Muzi, Marcin Pilipczuk, Paweł Rzążewski, and Manuel Sorge. Packing directed circuits quarter-integrally. Accepted to *27th Annual European Symposium on Algorithms, ESA 2019, September 9–13, 2019, Munich, Germany*, pages 72:1–72:13, 2019.
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3.3 Miscellaneous

[13] Radek Hušek, Tomáš Toufar, Dušan Knop, Tomáš Masařík, and Eduard Eiben. Steiner Tree Heuristics [PACE Challenge 2018, Track C]. *public repository*, `https://github.com/goderik01/PACE2018`, 2018.

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(excluding those already published)

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